Calculus II
MTH301

Virtual University of Pakistan
Knowledge beyond the boundaries

Table of Contents

Lecture No-1 Introduction

– Calculus is the mathematical tool used to analyze changes in physical quantities.
– Calculus is also Mathematics of Motion and Change.
– Where there is motion or growth, where variable forces are at work producing acceleration, Calculus is right mathematics to apply.

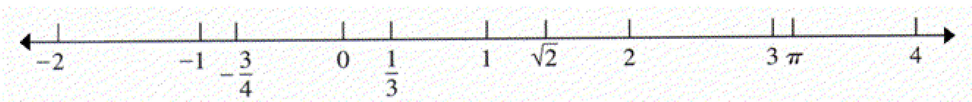
Differential Calculus **Deals with the Problem *of* Finding**

(1)Rate of change.
(2)Slope of curve.

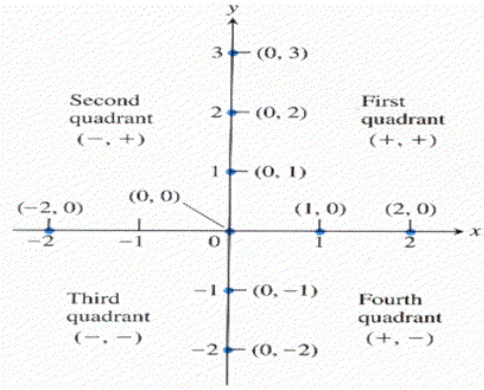
Velocities and acceleration of moving bodies. Firing angles that give cannons their maximum range. The times when planets would be closest together or farthest apart.

Integral Calculus
Deals with the Problem of determining a *Function* from information about its *rates of Change.*
Integral Calculus Enables Us
(1) To calculate lengths of curves.
(2) To find areas of irregular regions in plane.
(3) To find the volumes and masses of arbitrary solids
(4) To calculate the future location of a body from its present position and knowledge of the forces acting on it.

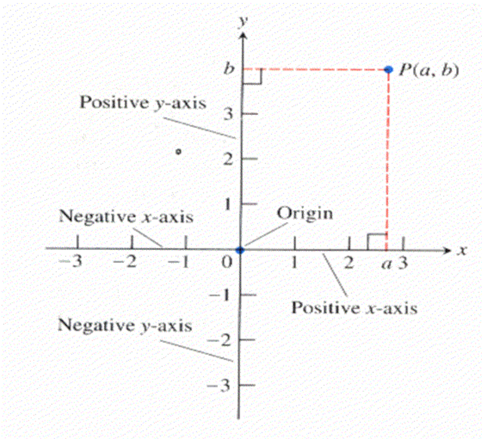
Reference Axis System
Before giving the concept of Reference Axis System we recall you the concept of real line and locate some points on the real line as shown in the figure below, also remember that the real number system consist of both Rational and Irrational numbers that is we can write set of real numbers as union of rational and irrational numbers.



Here in the above figure we have locate some of the rational as well as irrational numbers and also note that there are infinite real numbers between every two real numbers.
Now if you are working in two dimensions then you know that we take the two mutually perpendicular lines and call the horizontal line as x-axis and vertical line as y-axis and where these lines cut we take that point as origin.
Now any point on the x-axis will be denoted by an order pair whose first element which is also known as abscissa is a real number and other element of the order pair which is also known as ordinate will has 0 values.
Similarly any point on the y-axis can be representing by an order pair. Some points are shown in the figure below. Also note that these lines divide the plane into four regions, First ,Second ,Third and Fourth quadrants respectively. We take the positive real numbers at the right side of the origin and negative to the left side, in the case of x-axis. Similarly for y-axis and also shown in the figure.

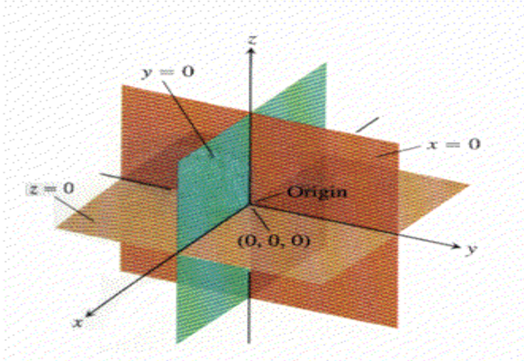


Location of a point
Now we will illustrate how to locate the point in the plane using x and y axis. Draw two perpendicular lines from the point whose position is to be determined. These lines will intersect at some point on the x-axis and y-axis and we can find out these points. Now the distance of the point of intersection of x-axis and perpendicular line from the origin is the X-C ordinate of the point P and similarly the distance from the origin to the point of intersection of y-axis and perpendicular line is the Y-coordinate of the point P as shown in the figure below.



Cartesian coordinates.

In space we have three mutually perpendicular lines as reference axis namely x ,y and z axis. Now you can see from the figure below that the planes x= 0 ,y=0 and z=0 divide the space into eight octants. Also note that in this case we have (0,0,0) as origin and any point in the space will have three coordinates.



The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

Sign of co-ordinates in different octants

First of all note that the equation x=0 represents a plane in the 3d space and in this plane every point has its x-coordinate as 0, also that plane passes through the origin as shown in the figure above. Similarly y=0 and z=0 are also define a plane in 3d space and have properties similar to that of x=0.Such that these planes also pass through the origin and any point in the plane y=0 will have y-coordinate as 0 and any point in the plane z=0 has z-coordinate as 0. Also remember that when two planes intersect we get the equation of a line and when two lines intersect then we get a plane containing these two lines. Now note that by the intersection of the planes x=0 and z=0 we get the line which is our y-axis. Also by the intersection of x=0 and y=0 we get the line which is z-axis, similarly you can easily see that by the intersection of z=0 and y=0 we get line which is x-axis.

Now these three planes divide the 3d space into eight octants depending on the positive and negative direction of axis.

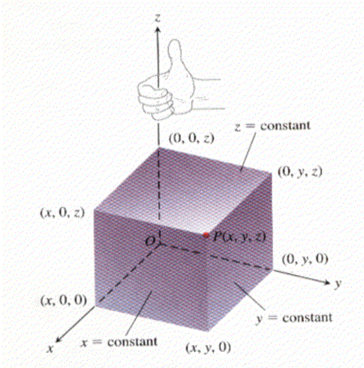
The octant in which every coordinate of any point has positive sign is known as first octant formed by the positive x, y and z –axis. Similarly in second octant every points has x-coordinate as negative and other two coordinates as positive correspond to negative x-axis and positive y and z axis.

Now one octant is that in which every point has x and y coordinate negative and z-coordinate positive, which is known as the third octant. Similarly we have eight octants depending on the sign of coordinates of a point. These are summarized below.

First octant	(+, +, +)	Formed by positive sides of the three axis.
Second octant	(-, +, +)	Formed by –ve x-axis and positive y and z-axis.
Third octant	(-, -, +)	Formed by –ve x and y axis with positive z-axis.
Fourth octant	(+, -, +)	Formed by +ve x and z axis and –ve y-axis.
Fifth octant	(+, +, -)	Formed by +ve x and y axis with -ve z-axis.
Sixth octant	(-, +, -)	Formed by –ve x and z axis with positive y-axis.
Seventh octant	(-, -, -)	Formed by –ve sides of three axis.
Eighth octant	(+, -, -)	Formed by -ve y and z-axis with +ve x-axis.

(Remember that we have two sides of any axis one of positive values and the other is of negative values)

Now as we told you that in space we have three mutually perpendicular lines as reference axis. So far you are familiar with the reference axis for 2d which consist of two perpendicular lines namely x-axis and y-axis. For the reference axis of 3d space we need another perpendicular axis which can be obtained by the cross product of the two vectors, now the direction of that vector can be obtained by Right hand rule. This is illustartaed below with diagram.



The Cartesian coordinate system is right-handed.

Concept of a Function

Historically, the term, function, denotes the dependence of one quantity on other quantity. The quantity x is called the independent variable and the quantity y is called the dependent variable. We write y = *f* (x) and we read y is a function of x.

The equation y = 2x defines y as a function of x because each value assigned to x determines unique value of y.

Examples of function

– The area of a circle depends on its radius r by the equation $A= \pi r^2$ so, we say that A is a function of r.

– The volume of a cube depends on the length of its side x by the equation $V= x^3$ so, we say that V is a function of x.

– The velocity *V* of a ball falling freely in the earth's gravitational field increases with time t until it hits the ground, so we say that V is function of t.

– In a bacteria culture, the number n of present after one day of growth depends on the number N of bacteria present initially, so we say that N is function of n.

Function of Several Variables

Many functions depend on more than one independent variable.

Examples

The area of a rectangle depends on its length *l* and width *w* by the equation

$A = l\, w$, so we say that *A* is a function of *l* and *w.*

The volume of a rectangular box depends on the length *l*, width *w* and height *h* by the equation

$V = l\, w\, h$ so, we say that *V* is a function of *l* , *w* and *h*.

The area of a triangle depends on its base length l and height h by the equation

A= ½ l h, so we say that A is a function of l and h.

The volume V of a right circular cylinder depends on its radius r and height h by the equation V= πr2h so, we say that V is a function of r and h.

Home Assignments:

In the first Lecture we recall some basic terminologies which are essential and prerequisite for this course. You can find the Home Assignments on the last page of Lecture # 1 at LMS.

Lecture No-2 Values of functions:

Consider the function $f(x) = 2x^2 - 1$, then $f(1) = 2(1)^2 - 1 = 1$, $f(4) = 2(4)^2 - 1 = 31$, $f(-2) = 2(-2)^2 - 1 = 7$

$f(t-4) = 2(t-4)^2 - 1 = 2t^2 - 16t + 31$

These are the values of the function at some points.

Example

Now we will consider a function of two variables, so consider the function $f(x,y) = x^2y+1$ then $f(2,1) = (2^2)1+1=5$, $f(1,2) = (1^2)2+1=3$, $f(0,0) = (0^2)0+1=1$, $f(1,-3) = (1^2)(-3)+1=-2$, $f(3a,a) = (3a)^2a+1=9a^3+1$, $f(ab,a-b) = (ab)^2(a-b)+1=a^3b^2-a^2b^3+1$

These are values of the function at some points.

Example:

Now consider the function $f(x,y) = x + \sqrt[3]{xy}$ then

(a) $f(2,4) = 2 + \sqrt[3]{(2)(4)} = 2 + \sqrt[3]{8} = 2 + 2 = 4$

(b) $f(t,t^2) = t + \sqrt[3]{(t)(t^2)} = t + \sqrt[3]{t^3} = t + t = 2t$

(c) $f(x,x^2) = x + \sqrt[3]{(x)(x^2)} = x + \sqrt[3]{x^3} = x + x = 2x$

(d) $f(2y^2,4y) = 2y^2 + \sqrt[3]{(2y^2)(4y)} = 2y^2 + \sqrt[3]{8y^3} = 2y^2 + 2y$

Example:

Now again we take another function of three variables

$$f(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2} \text{ Then}$$

$$f(0,\frac{1}{2},\frac{1}{2}) = \sqrt{1 - 0 - (\frac{1}{2})^2 - (\frac{1}{2})^2} = \sqrt{\frac{1}{2}}$$

Example:

Consider the function $f(x,y,z) = xy^2z^3+3$ then at certain points we have

$f(2,1,2) = (2)(1)^2(2)^3+3=19$, $f(0,0,0) = (0)(0)^2(0)^3+3=3$, $f(a,a,a) = (a)(a)^2(a)^3+3=a^6+3$
$f(t,t^2,-t) = (t)(t^2)^2(-t)^3+3=-t^8+3$, $f(-3,1,1) = (-3)(1)^2(1)^3+3=0$

Example:

Consider the function $f(x,y,z) = x^2y^2z^4$ where $x(t) = t^3$ $y(t) = t^2$ and $z(t) = t$

(a) $f(x(t),y(t),z(t)) = [x(t)]^2[y(t)]^2[z(t)]^4 = [t^3]^2[t^2]^2[t]^4 = t^{14}$

(b) $f(x(0),y(0),z(0)) = [x(0)]^2[y(0)]^2[z(0)]^4 = [0^3]^2[0^2]^2[0]^4 = 0$

Example:

Let us consider the function $f(x,y,z) = xyz + x$ then

$$f(xy,y/x,xz) = (xy)(y/x)(xz) + xy = xy^2z+xy.$$

Example:

Let us consider $g(x,y,z) = z\ \mathrm{Sin}(xy)$, $u(x,y,z) = x^2z^3$, $v(x,y,z) = Pxyz$, $w(x,y,z) = \frac{xy}{z}$ Then.

$g(u(x,y,z), v(x,y,z), w(x,y,z)) = w(x,y,z)\ \mathrm{Sin}(u(x,y,z)\ v(x,y,z))$

Now by putting the values of these functions from the above equations we get

$g(u(x,y,z), v(x,y,z), w(x,y,z)) = \frac{xy}{z}\ \mathrm{Sin}[(x^2z^3)(Pxyz)] = \frac{xy}{z}\ \mathrm{Sin}[(Pyx^3z^4)]$.

Example:

Consider the function $g(x,y) = y\,\text{Sin}(x^2 y)$ and $u(x,y) = x^2y^3$ $v(x,y) = \pi\, xy$ Then

$$g(u(x,y), v(x,y)) = v(x,y)\, \text{Sin}([u(x,y)]^2\, v(x,y))$$

By putting the values of these functions we get

$$g(u(x,y), v(x,y)) = \pi\, xy\, \text{Sin}([x^2y^3]^2\ \pi\, xy) = \pi\, xy\, \text{Sin}(x^5y^7).$$

Function of One Variable

A function f of one real variable x is a rule that assigns a unique real number f(x) to each point x in some set D of the real line.

Function of two Variables

A function f in two real variables x and y, is a rule that assigns unique real number f (x,y) to each point (x,y) in some set D of the xy-plane.

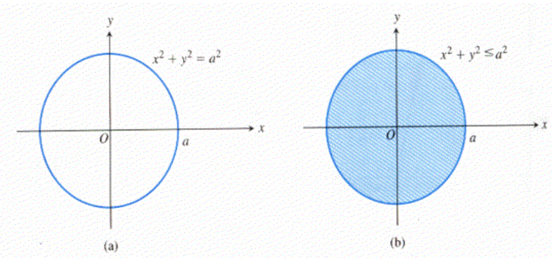
Function of three variables:

A function f in three real variables x, y and z, is a rule that assigns a unique real number f (x,y,z) to each point (x,y,z) in some set D of three dimensional space.

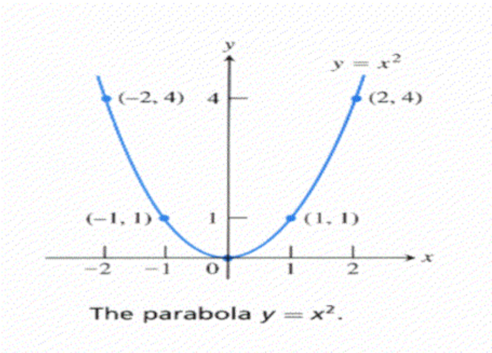
Function of n variables:

A function f in n variable real variables $x_1, x_2, x_3, \ldots\ldots, x_n$, is a rule that assigns a unique real number $w = f(x_1, x_2, x_3, \ldots\ldots, x_n)$ to each point $(x_1, x_2, x_3, \ldots\ldots, x_n)$ I n some set D of n dimensional space.

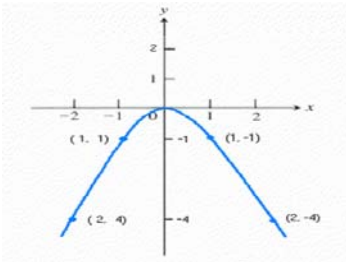
Circles and Disks:



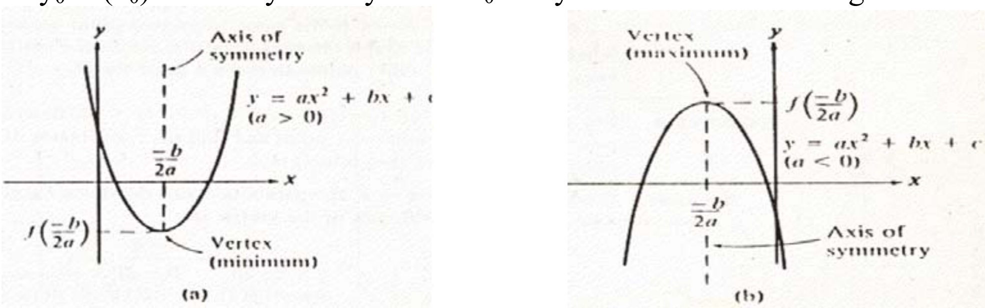
PARABOLA



Parabola $y = -x^2$



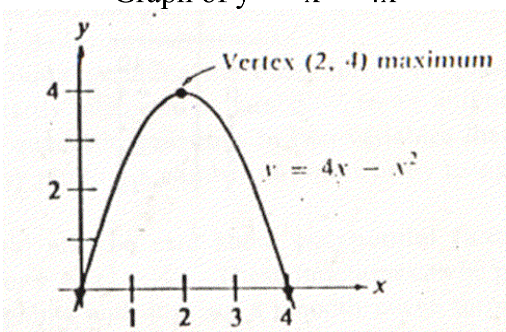
General equation of the Parabola opening upward or downward is of the form
$y = f(x) = ax^2+bx + c$. Opening upward if $a > 0$. Opening downward if $a < 0$.
x co-ordinate of the vertex is given by $x_0 = -b/2a$. So the y co-ordinate of the vertex is $y_0= f(x_0)$ axis of symmetry is $x = x_0$. As you can see from the figure below



Sketching of the graph of parabola $y = ax^2+bx + c$
Finding vertex: x – co-ordinate of the vertex is given by $x_0= - b/2a$
So, y – co-ordinate of the vertex is $y_0= a\, x_0^2+b\, x_0 + c$. Hence vertex is $V(x_0 , y_0)$.

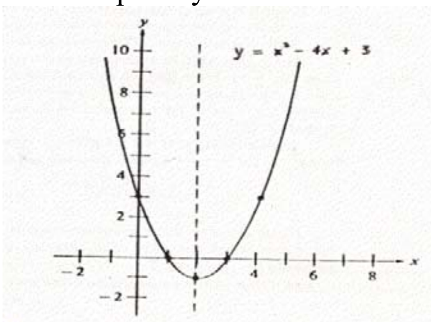
Example: Sketch the parabola $y = - x^2 + 4x$
Solution: Since $a = -1 < 0$, parabola is opening downward. Vertex occurs at
$x = - b/2a = (-4)/2(-1) =2$. Axis of symmetry is the vertical line $x = 2$. The y-co-ordinate of the vertex is$y = -(2)^2 + 4(2) = 4$. Hence vertex is V(2 , 4). The zeros of the parabola (i.e. the point where the parabola meets x-axis) are the solutions to $-x^2 +4x = 0$ so $x = 0$ and $x = 4$. Therefore (0,0)and (4,0) lie on the parabola. Also (1,3) and (3,3) lie on the parabola.

Graph of $y = - x^2 + 4x$

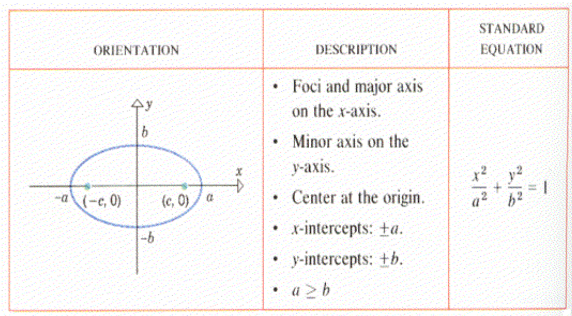


Example $y = x^2 - 4x+3$
Solution: Since $a = 1 > 0$, parabola is opening upward.Vertex occurs at
$x = - b/2a = (4)/2 =2$.Axis of symmetry is the vertical line $x = 2$. The y co-ordinate of the vertex is $y = (2)^2 - 4(2) + 3 = -1$.Hence vertex is V(2 , -1)The zeros of the parabola (i.e. the point where the parabola meets x-axis) are the solutions to $x^2 - 4x + 3 = 0$, so $x = 1$ and $x = 3$.Therefore (1,0)and (3,0) lie on the parabola. Also (0 ,3) and (4, 3) lie on the parabola.

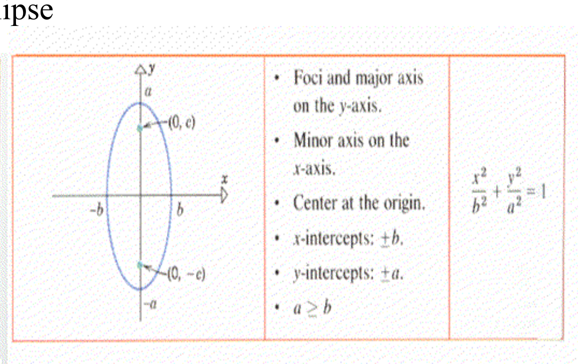
Graph of y = $x^2 - 4x+3$



Ellipse

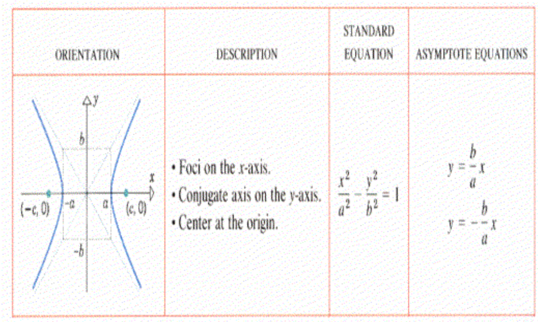


ORIENTATION	DESCRIPTION	STANDARD EQUATION
$-a$, $(-c, 0)$, $(c, 0)$, a, b, $-b$	• Foci and major axis on the x-axis. • Minor axis on the y-axis. • Center at the origin. • x-intercepts: $\pm a$. • y-intercepts: $\pm b$. • $a \geq b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



a, $(0, c)$, $-b$, b, $(0, -c)$, $-a$	• Foci and major axis on the y-axis. • Minor axis on the x-axis. • Center at the origin. • x-intercepts: $\pm b$. • y-intercepts: $\pm a$. • $a \geq b$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

Hyperbola



ORIENTATION	DESCRIPTION	STANDARD EQUATION	ASYMPTOTE EQUATIONS
b, $(-c, 0)$, $-a$, a, $(c, 0)$, $-b$	• Foci on the x-axis. • Conjugate axis on the y-axis. • Center at the origin.	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$y=\frac{b}{a}x$ $y=-\frac{b}{a}x$

Home Assignments:

In this lecture we recall some basic geometrical concepts which are prerequisite for this course and you can find all these concepts in the chapter # 12 of your book Calculus By Howard Anton.

Lecture No-3 Elements of three dimensional geometry

Distance formula in three dimension

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points such that PQ is not parallel to one of the coordinate axis Then $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ Which is known as Distance fromula between the points P and Q.

Example of distance formula

Let us considerthe points $A\,(3, 2, 4)$, $B\,(6, 10, -1)$,
and $C\,(9, 4, 1)$
Then

$$|AB| = \sqrt{(6-3)^2 + (10-2)^2 + (-1-4)^2} = \sqrt{98} = 7\sqrt{2}$$
$$|AC| = \sqrt{(9-3)^2 + (4-2)^2 + (1-4)^2} = \sqrt{49} = 7$$
$$|BC| = \sqrt{(9-6)^2 + (4-10)^2 + (1+1)^2} = \sqrt{49} = 7$$

Mid point of two points

If R is the middle point of the line segment PQ, then the co-ordinates of the middle points are
x= (x1+x2)/2 ,
y= (y1+y2)/2 ,
z= (z1+z2)/2

Let us consider tow points A(3,2,4) and B(6,10,-1)
Then the co-ordinates of mid point of AB is

[(3+6)/2,(2+10)/2,(4-1)/2]
= (9/2,6,3/2)

Direction Angles

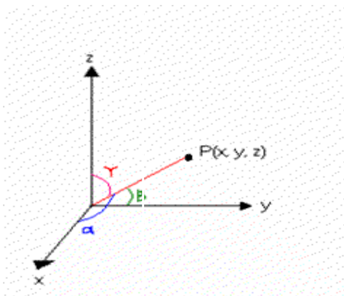
The**direction angles** $\alpha\ \beta, \gamma$ of a lineare defined as

α = Angle between lineand the positive x-axis

β = Angle between line and the positive y-axis

γ = Angle between lineand the positive $z$axis.

By definition, each of these angles lies between 0 and π

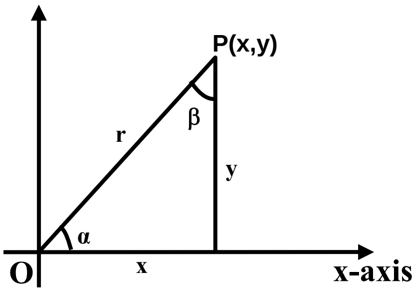


Direction Ratios

Cosines of direction angles are called direction cosines
Any multiple of direction cosines are called direction numbers or direction ratios of the line L.

Given a point, finding its Direction cosines

y-axis

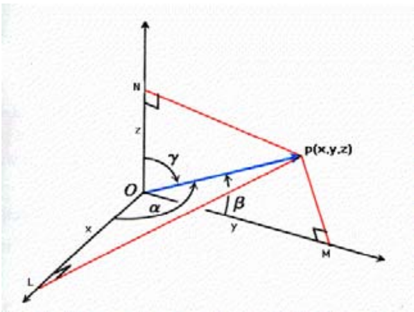


From triangle we can write

cos α = **x/r**

cos β = **y/r**

Direction angles of a Line



The angles which a line makes with positive x,y and z-axis are known as Direction Angles. In the above figure the blue line has direction angles as α,□and□which are the angles which blue line makes with x,y and z-axis respectively.

Direction cosines:

Now if we take the cosine of the Direction Angles of a line then we get the Direction cosines of that line. So the Direction Cosines of the above line are given by

$$\cos\alpha = \frac{x}{OP} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\beta = \frac{y}{OP} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

Similarly,

$$\cos\gamma = \frac{z}{OP} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Direction cosines and direction ratios of a line joining two points

•For a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, x_2)$ the direction ratios are

$x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the directions cosines are $\dfrac{x_2 - x_1}{|PQ|}, \dfrac{y_2 - y_1}{|PQ|}$ *and* $\dfrac{z_2 - z_1}{|PQ|}$.

Example For a line joining two points P(1,3,2) and Q(7,-2,3) the direction ratios are

7 - 1, -2 – 3 , 3 – 2

6 , -5 , 1

and the directions cosines are

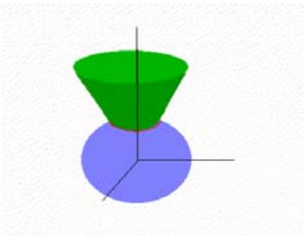
6/√62 , -5/√62 , 1/ √62

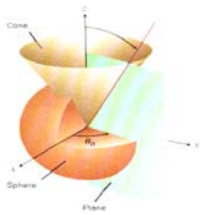
In two dimensional space the graph of an equation relating the variables x and y is the set of all point (x, y) whose co-ordinates satisfy the equation. Usually, such graphs are curves. In three dimensional space the graph of an equation relating the variables x, y and z is the set of all point (x, y, z) whose co-ordinates satisfy the equation. Usually, such graphs are surfaces.

Intersection of two surfaces

•Intersection of two surfaces is a curve in three dimensional space.
•It is the reason that a curve in three dimensional space is represented by two equations representing the intersecting surfaces.

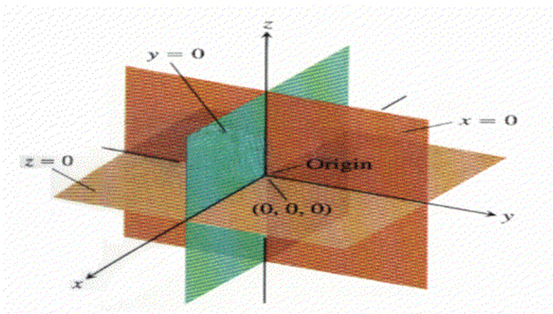
Intersection of Cone and Sphere





Intersection of Two Planes

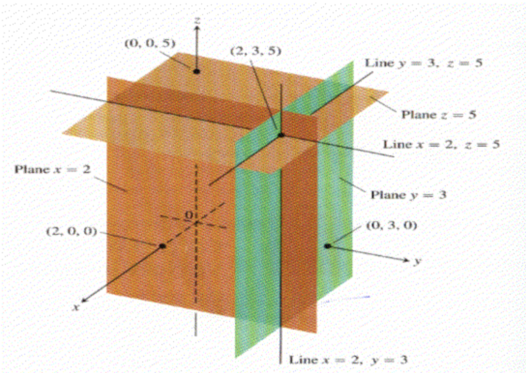
If the two planes are not parallel, then they intersect and their intersection is a straight line. Thus, two non-parallel planes represent a straight line given by two simultaneous linear equations in x, y and z and are known as non-symmetric form of equations of a straight line.



The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

REGION	DESCRIPTION	EQUATION
xy-plane	Consists of all points of the form (x, y, 0)	z = 0
xz-plane	Consists of all points of the form (x, 0, z)	y = 0
yz-plane	Consists of all points of the form (0, y, z)	x = 0
x-axis	Consists of all points of the form (x, 0, 0)	y = 0, z = 0
y-axis	Consists of all points of the form (0, y, 0)	z = 0, x = 0
z-axis	Consists of all points of the form (0, 0, z)	x = 0, y = 0

Planes parallel to Co-ordinate Planes



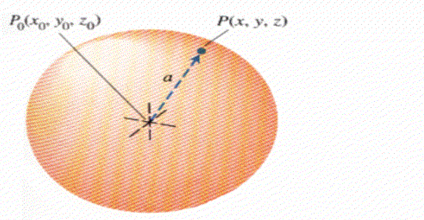
General Equation of Plane

Any equation of the form

$$ax + by + cz + d = 0$$

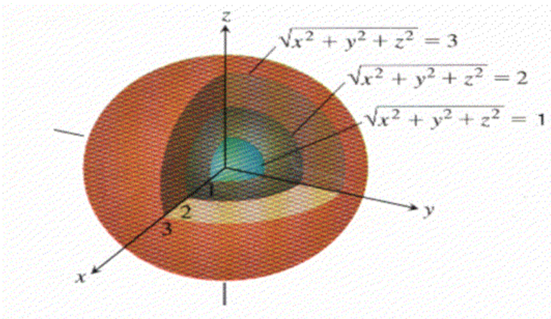
where a, b, c, d are real numbers,represent a plane.

Sphere



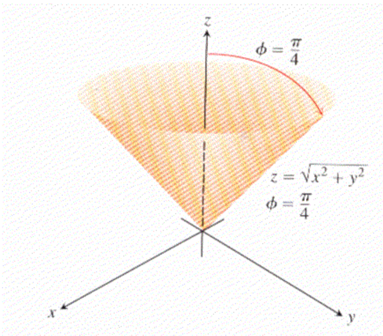
The standard equation of the sphere of radius a centered at (x_0, y_0, z_0) is

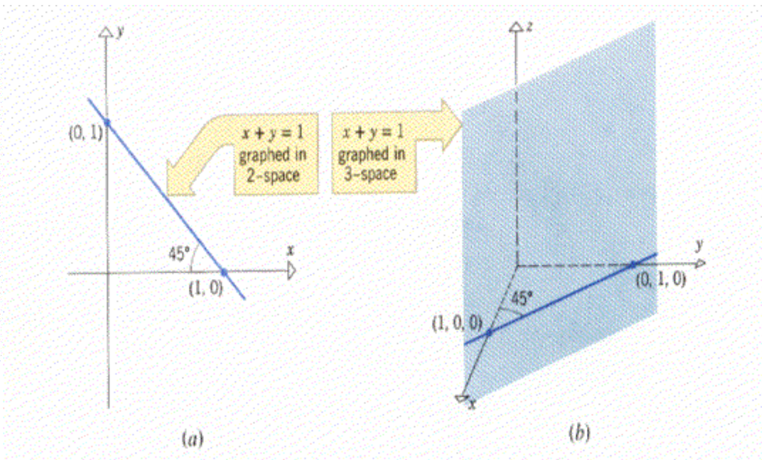
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

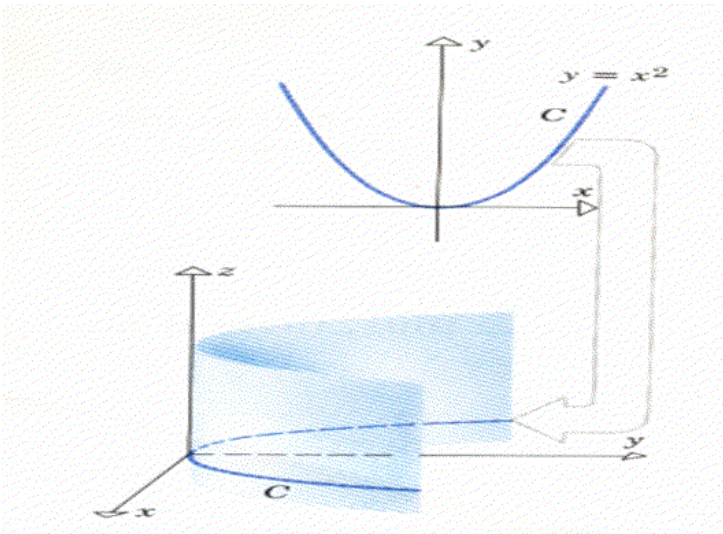


The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres.

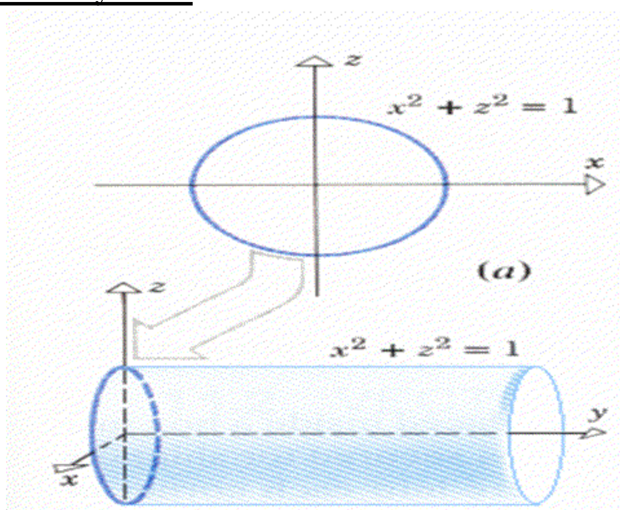
Right Circular Cone



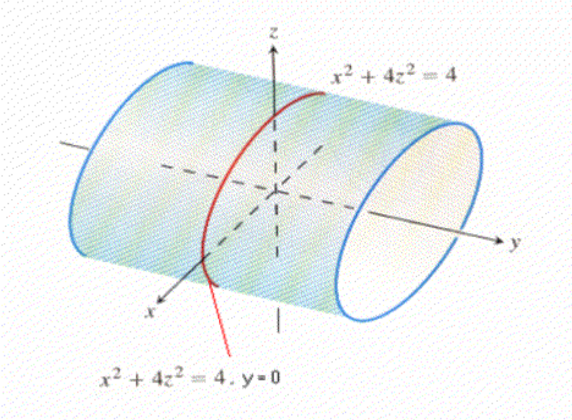


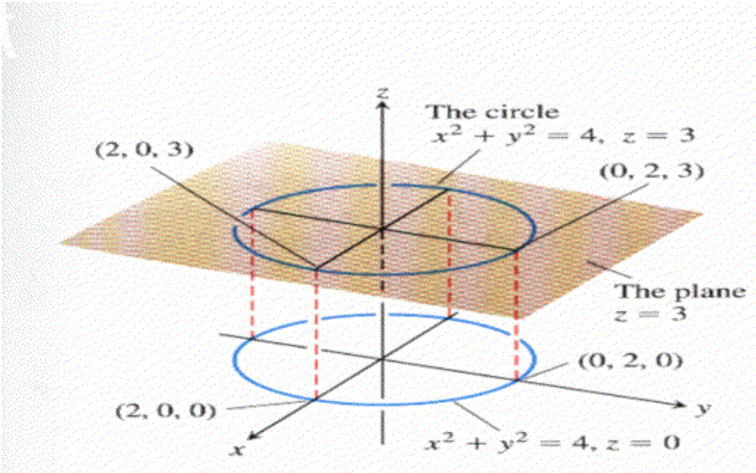


Horizontal Circular Cylinder



Horizontal Elliptic Cylinder





The circle $x^2 + y^2 = 4$, $z = 3$.

Overview of Lecture # 3

Chapter # 14

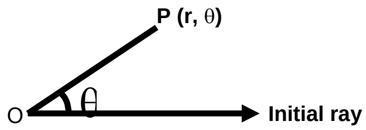
Three Diamentional Space

Page # 657

Book **CALCULUS by HOWARD ANTON**

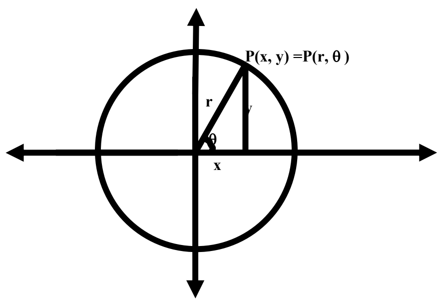
Lecture -4 Polar co-ordinates

You know that position of any point in the plane can be obtained by the two perpendicular lines known as x and y axis and together we call it as Cartesian coordinates for plane. Beside this coordinate system we have another coordinate system which can also use for obtaining the position of any point in the plane. In that coordinate system we represent position of each particle in the plane by "r" and "θ "where "r" is the distance from a fixed point known as pole and θ is the measure of the angle.



"O" is known as pole.

Conversion formula from polar to Cartesian coordinates and vice versa



From above diagram and remembering the trigonometric ratios we can write $x = r \cos \theta$, $y = r \sin \theta$. Now squaring these two equations and adding we get,

$$x^2 + y^2 = r^2$$

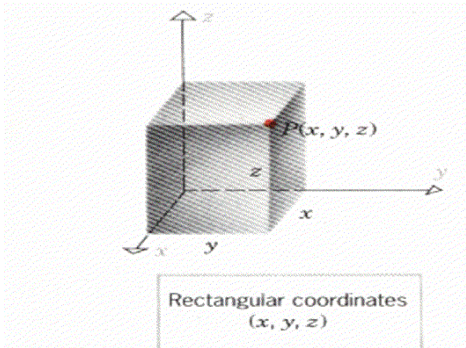
Dividing these equations we get

$$y/x = \tan\theta$$

These two equations gives the relation between the Plane polar and Plane Cartesian coordinates.

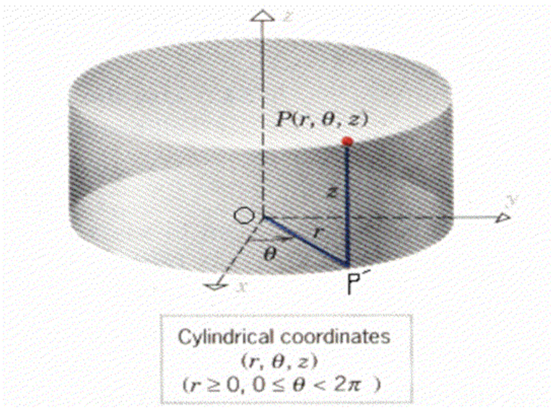
Rectangular co-ordinates for 3d

Since you know that the position of any point in the 3d can be obtained by the three mutually perpendicular lines known as x ,y and z – axis and also shown in figure below, these coordinate axis are known as Rectangular coordinate system.



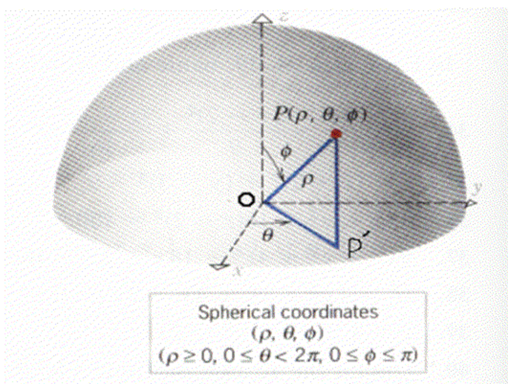
Cylindrical co-ordinates

Beside the Rectangular coordinate system we have another coordinate system which is used for getting the position of the any particle is in space known as the cylindrical coordinate system as shown in the figure below.



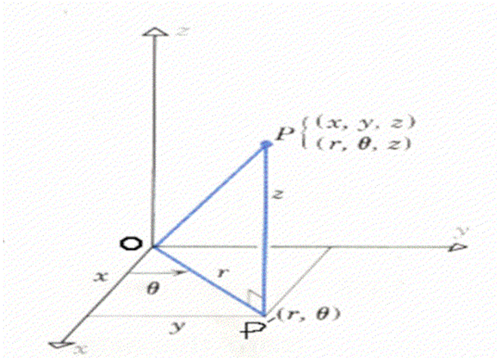
Spherical co-ordinates

Beside the Rectangular and Cylindrical coordinate systems we have another coordinate system which is used for getting the position of the any particle is in space known as the spherical coordinate system as shown in the figure below.



Conversion formulas between rectangular and cylindrical co-ordinates

Now we will find out the relation between the Rectangular coordinate system and Cylindrical coordinates. For this consider any point in the space and consider the position of this point in both the axis as shown in the figure below.



In the figure we have the projection of the point P in the xy-Plane and write its position in plane polar coordinates and also represent the angle θ now from that projection we draw perpendicular to both of the axis and using the trigonometric ratios find out the following relations.

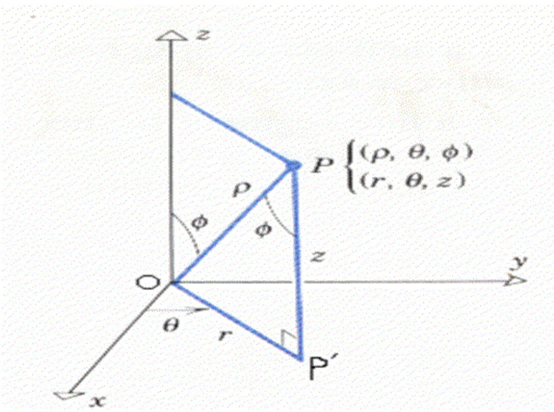
$(r, \theta, z) \to (x, y, z)$

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$

Conversion formulas between cylindrical and spherical co-ordinates

Now we will find out the relation between spherical coordinate system and Cylindrical coordinate system. For this consider any point in the space and consider the position of this point in both the axis as shown in the figure below.



First we will find the relation between Planes polar to spherical, from the above figure you can easily see that from the two right angled triangles we have the following relations.

$(\rho, \theta, \phi) \to (r, \theta, z)$

$r = \rho \sin\phi\,,\ \ \theta = \theta,\ \ z = \rho\cos\phi$

Now from these equations we will solve the first and second equation for ρ and ϕ. Thus we have

$(r,\ \theta, z) \rightarrow (\rho,\theta,\phi)$

$\rho = \sqrt{r^2+z^2}\ \ \theta = \theta,\ \ \tan\phi = \frac{r}{z}$

Conversion formulas between rectangular and spherical co-ordinates

$$\boldsymbol{(\rho,\ \theta,\ \Phi) \rightarrow (x,\ y,\ z)}$$

Since we know that the relation between Cartesian coordinates and Polar coordinates are

$x = r\cos\theta$, $y = r\sin\theta$ and $z = z$ .We also know the relation between Spherical and cylindrical coordinates are,

$r = \rho \sin\phi\,,\ \ \theta = \theta,\ \ z = \rho\cos\phi$

Now putting this value of "r" and "z" in the above formulas we get the relation between spherical coordinate system and Cartesian coordinate system. Now we will find

$$(x, y, z) \rightarrow (\rho, \theta, \Phi)$$

$$\begin{aligned} x^2 + y^2 + z^2 &= (\rho\sin\Phi\cos\theta)^2 + (\rho\sin\Phi\sin\theta)^2 + (\rho\cos\Phi)^2 \\ &= \rho^2\{\sin^2\Phi(\cos^2\theta + \sin^2\theta) + \cos^2\Phi)\} \\ &= \rho^2(\sin^2\Phi + \cos^2\Phi)^2 = \rho^2 \end{aligned}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

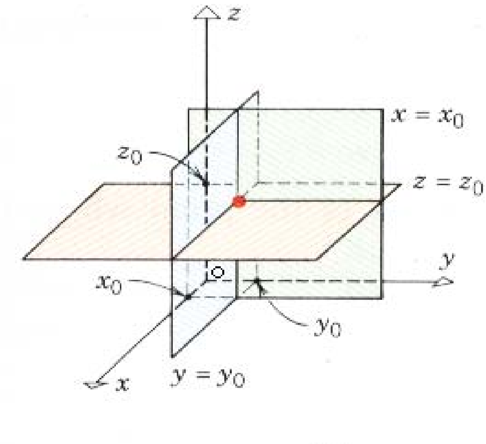
$\tan\theta = y/x$ and $\cos\Phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

Constant surfaces in rectangular co-ordinates

The surfaces represented by equations of the form

$$\mathbf{x = x_0,\ y = y_0,\ z = z_0}$$

where x_o, y_o, z_o are constants, are planes parallel to the xy-plane, xz-plane and xy-plane, respectively. Also shown in the figure



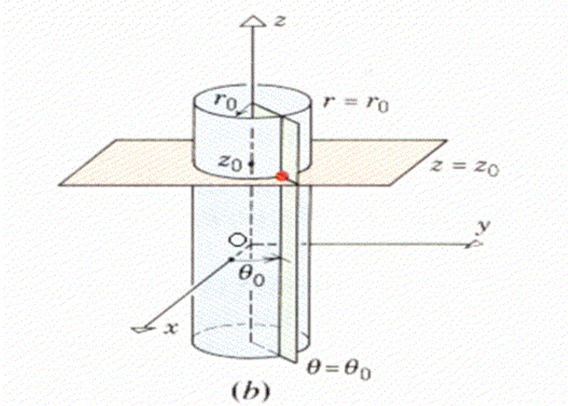
(a)

Constant surfaces in cylindrical co-ordinates

The surface **$r = r_0$ is a right cylinder** of radius r_0 centered on the z-axis. At each point (r, θ, z) this surface on this cylinder, r has the value r_0 , z is unrestricted and

$$0 \leq \theta < 2\pi.$$

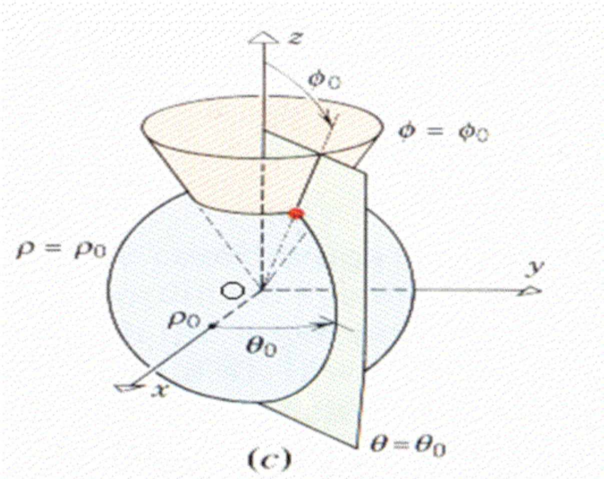
The surface **$\theta = \theta_0$ is a half plane** attached along the z-axis and making angle θ_0 with the positive x-axis. At each point (r, θ, z) on the surface ,θ has the value θ_0, z is unrestricted and r ≥0. The surfaces **$z = z_0$ is a horizontal plane.** At each point (r, θ, z) this surface z has the value z_0 , but r and θ are unrestricted as shown in the figure below.



(b)

Constant surfaces in spherical co-ordinates

The surface **$\rho = \rho_0$ consists of all points whose distance ρ from origin is ρ_0.** Assuming that ρ_0 to be nonnegative, this is a sphere of radius ρ_0 centered at the origin. The surface **$\theta = \theta_0$ is a half plane** attached along the z-axis and making angle θ_0 with the positive x-axis. The surface $\Phi = \Phi_0$ consists of all points from which a line segment to the origin makes an angle of Φ_0 with the positive z-axis. Depending on whether $0 < \Phi_0 < \pi/2$ or $\pi/2 < \Phi_0 < \pi$, this will be a cone opening up or opening down. If $\Phi_0 = \pi/2$, then the cone is flat and the surface is the xy-plane.



Spherical Co-ordinates in Navigation

Spherical co-ordinates are related to longitude and latitude coordinates used in navigation. Let us consider a right handed rectangular coordinate system with origin at earth's center, positive z-axis passing through the north pole,and x-axis passing through the prime meridian. Considering earth to be a perfect sphere of radius ρ = 4000 miles, then each point has spherical coordinates of the form (4000,θ,Φ) where Φ and θ determine the latitude and longitude of the point. Longitude is specified in degree east or west of the prime meridian and latitudes is specified in degree north or south of the equator.

Domain of the Function

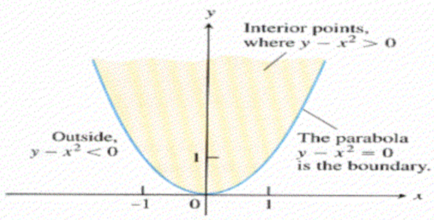
- In the above definitions the set D is the domain of the function.
- The Set of all values which the function assigns for every element of the domain is called the Range of the function.
- When the range consist of real numbers the functions are called the real valued function.

NATURAL DOMAIN

Natural domain consists of all points at which the formula has no divisions by zero and produces only real numbers.

Examples

Consider the Function $\varpi = \sqrt{y - x^2}$. Then the domain of the function is
$y \geq x^2$ Which can be shown in the plane as



The domain of $f(x, y) = \sqrt{y - x^2}$ consists of the shaded region and its bounding parabola $y = x^2$.

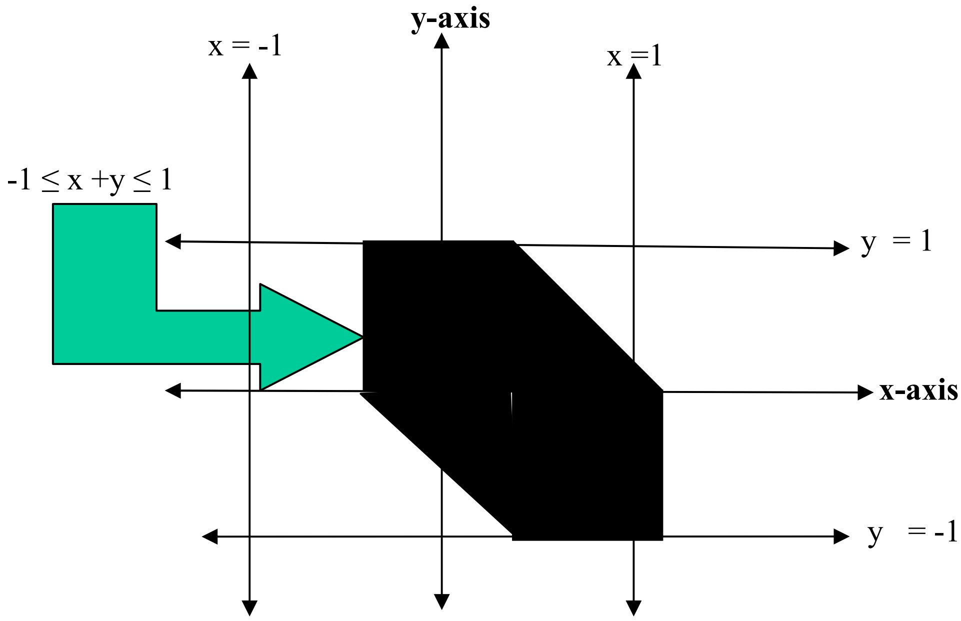
and Range of the function is $[0, \infty)$.

Domain of function w = 1/xy is the whole xy- plane Excluding x-axis and y-axis, because at x and y axis all the points has x and y coordinates as 0 and thus the defining formula for the function gives us 1/0. So we exclude them.

Lecture No-5 Limit of Multivariable Function

$f(x,y) = \sin^{-1}(x+y)$

Domain of f is the region in which $-1 \leq x + y \leq 1$



Domains and Ranges

Functions	Domain	Range
$\omega = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$\omega = \dfrac{1}{x^2 + y^2 + z^2}$	$(x,y,z) \neq (0, 0, 0)$	$(0, \infty)$
$\omega = xy \ln z$	Half space $z > 0$	$(-\infty, \infty)$

Examples of domain of a function

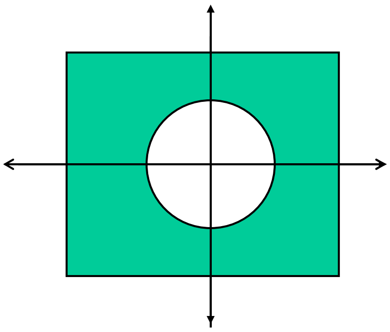
$f(x, y) = xy\sqrt{y - 1}$ Domain of f consists of region in xy plane where $y \geq 1$

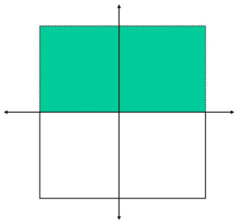
$f(x, y) = \sqrt{x^2 + y^2 - 4}$

Domain of f consists of region

in xy plane where $x^2 + y^2 \geq 4$

As shown in the figure



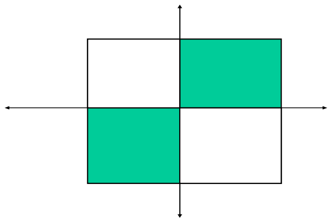


$f(x, y) = \ln xy$

Domain of *f* consists of region lying in first and third quadrants in xy plane as shown in above figure right side.

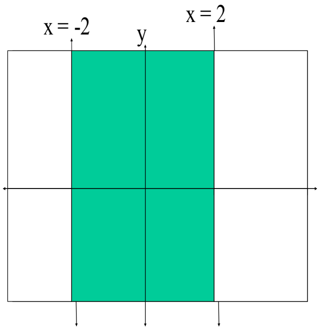
$f(x, y, z) = e^{xyz}$

Domain of f consists of region of three dimensional space



$$f(x, y) = \frac{\sqrt{4 - x^2}}{y^2 + 3}$$

Domain of f consists of region in xy plane $x^2 \le 4$, $-2 \le x \le 2$



$$f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$$

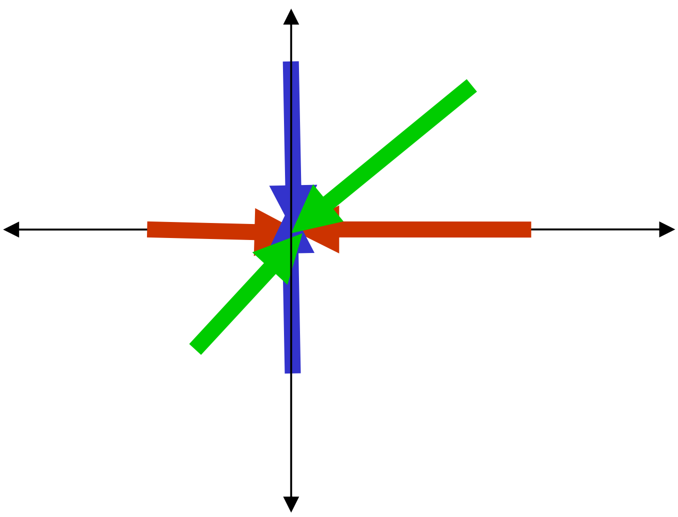
Domain of f consists of region in three dimensional space occupied by sphere centre at (0, 0, 0) and radius 5.

$$f(x, y) = \frac{x^3 + 2x^2y - xy - 2y^2}{x + 2y}$$

f(0,0) is not defined but we see that limit exits.

Approaching to (0,0) through x-axis	f (x,y)	Approaching to (0,0) through y-axis	f (x,y)
(0.5,0)	0.25	(0,0.1)	-0.1
(0.25,0)	0.0625	(0,0.001)	-0.001
(0.1,0)	0.01	(0,0.00001)	0.00001
(-0.25,0)	0.0625	(0,-0.001)	0.001
(-0.1,0)	0.01	(0,-0.00001)	0.00001

Approaching to (0,0) through y = x	f (x,y)
(0.5,0.5)	-0.25
(0.1,0.1)	-0.09
(0.01,0.01)	-0.0099
(-0.5,-0.5)	0.75
(-0.1,-0.1)	0.11
(-0.01,-0.01)	0.0101



Example

$$f(x,y) = \frac{xy}{x^2+y^2}$$

f(0,0) is not defined and we see that limit also does not exit.

Approaching to (0,0) through x-axis (y = 0)	f (x,y)	Approaching to (0,0) through y = x	f (x,y)
(0.5,0)	0	(0.5,0.5)	0.5
(0.1,0)	0	(0.25,0.25)	0.5
(0.01,0)	0	(0.1,0.1)	0.5
(0.001,0)	0	(0.05,0.05)	0.5
(0.0001,0)	0	(0.001,0.001)	0.5
(-0.5,0)	0	(-0.5,-0.5)	0.5
(-0.1,0)	0	(-0.25,-0.25)	0.5
(-0.01,0)	0	(-0.1,-0.1)	0.5
(-0.001,0)	0	(-0.05,-0.05)	0.5
(-0.0001,0)	0	(-0.001,-0.001)	0.5

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0 \text{ (along } y = 0)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.5 \text{ (along } y = x)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist.}$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Let (x, y) approach $(0, 0)$ along the line $y = x$.

$$f(x, y) = \frac{xy}{x^2+y^2} = \frac{x \cdot x}{x^2+x^2} = \frac{1}{1+1} \qquad x \neq 0.$$

$$= \frac{1}{2}$$

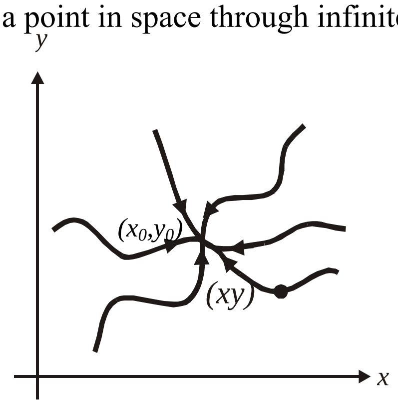
Let (x, y) approach $(0, 0)$ along the line $y = 0$.

$$f(x, y) = \frac{x.(0)}{x^2 + (0)^2} = 0 \ , \quad x \neq 0.$$

Thus $f(x, y)$ assumes two different values as (x,y) approaches (0,0) along two different paths.

$\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

We can approach a point in space through infinite paths some of them are shown in the figure below.



<u>Rule for Non-Existence of a Limit</u>

If in

$$\lim_{(x, y) \to (a, b)} f(x, y)$$

We get <u>*two or more different values as we approach (a, b) along different paths*</u>, then

$$\lim_{(x, y) \to (a, b)} f(x, y)$$

does not exist.

The paths along which (a, b) is approached may be <u>*straight lines or plane curves through (a, b).*</u>

<u>Example</u>

$$\lim_{(x, y) \to (2, 1)} \frac{x^3 + x^2 y - x - y^2}{x + 2y}$$

$$= \frac{\lim_{(x, y) \to (2, 1)} (x^3 + 2x^2 y - xy - 2y^2)}{\lim_{(x, y) \to (2, 1)} (x + 2y)}$$

$$= \frac{\lim\limits_{(x,\,y) \to (2,\,1)} (x^3 + 2\ x^2\ y - x\ -\ y^2)}{\lim\limits_{(x,\,y) \to (2,\,1)} (x + 2y)}$$

Example

$$\lim_{(x,\,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

We set

$$x = r\cos\theta,\ \ y = r\sin\theta$$

then $\dfrac{x}{\sqrt{x^2 + y^2}} = \dfrac{r\cos\theta \,.\, r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$

$= \ r\ \cos\theta\ \sin\theta$, for $r > 0$

Since $r = \sqrt{x^2 + y^2}$, $r \to 0$ as $(x, y) \to (0, 0)$,

$$\lim_{(x,\,y) \to (0,\,0)} \frac{x}{\sqrt{x^2 + y^2}} = \lim_{r \to 0} r\cos\theta\sin\theta = 0,$$

since $|\cos\theta\sin\theta| \le 1$ **for all value of** θ.

RULES FOR LIMIT

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ *and* $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$ Then

(a) $\lim\limits_{(x,y)\to(x_0,y_0)} cf(x,y) = cL_1$ (if c is constant)

(b) $\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y) + g(x,y)\} = L_1 + L_2$

$\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y) - g(x,y)\} = L_1 - L_2$

(d) $\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y)g(x,y)\} = L_1 L_2$

(e) $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$ (if $L_2 = 0$)

$\lim\limits_{(x,y)\to(x_0,y_0)} c = c$ (c a constant), $\lim\limits_{(x,y)\to(x_0,y_0)} x_0 = x_0$, $\lim\limits_{(x,y)\to(x_0,y_0)} y_0 = y_0$

Similarly for the function of three variables.

Overview of lecture# 5

In this lecture we recall you all the limit concept which are prerequisite for this course and you can find all these concepts in the chapter # 16 (topic # 16.2)of your Calculus By Howard Anton.

Lecture No -6 Geometry of continuous functions

Geometry of continuous functions in one variable or Informal definition of continuity of function of one variable.

A function is continuous if we draw its graph by a pen then the pen is not raised so that there is no gap in the graph of the function

Geometry of continuous functions in two variables or Informal definition of continuity of function of two variables.

The graph of a continuous function of two variables to be constructed from a thin sheet of clay that has been hollowed and pinched into peaks and valleys without creating tears or pinholes.

Continuity of functions of two variables

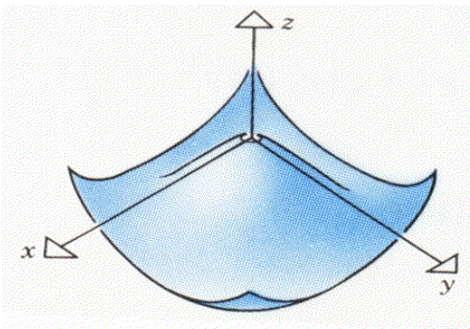
A function f of two variables is called continuous at the point (x_0, y_0) if

1. $f(x_0, y_0)$ if defined.
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

The requirement that $f(x_0, y_0)$ must be defined at the point (x_0, y_0) eliminates the possibility of a hole in the surface $z = f(x_0, y_0)$ above the point (x_0, y_0).

Justification of three points involving in the definition of continuity.

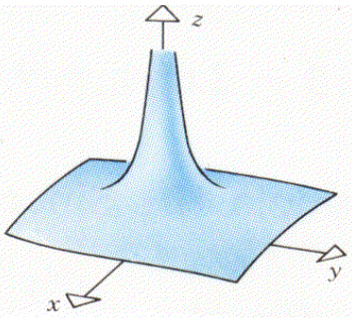
(1) Consider the function of two variables $x^2 + y^2 \ln(x^2 + y^2)$ now as we know that the Log function is not defined at 0, it means that when x = 0 and y = 0 our function $x^2 + y^2 \ln(x^2 + y^2)$ is not defined. Consequently the surface $z = x^2 + y^2 \ln(x^2 + y^2)$ will have a hole just above the point (0,0)as shown in the graph of $x^2 + y^2 \ln(x^2 + y^2)$



$z = (x^2 + y^2)\ln(x^2 + y^2)$ has a hole at the origin.

(2) The requirement that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists ensures us that the surface z = f(x,y) of the function f(x,y) doesn't become infinite at (x_0, y_0) or doesn't oscillate widely.

Consider the function of two variables $\frac{1}{\sqrt{x^2+y^2}}$ now as we know that the Natural domain of the function is whole the plane except origin. Because at origin we have x = 0 and y =0 in the defining formula of the function we will have at that point 1/0 which is infinity. Thus the limit of the function $\frac{1}{\sqrt{x^2+y^2}}$ does not exists at origin. Consequently the surface $z = \frac{1}{\sqrt{x^2+y^2}}$ will approaches towards infinity when we approaches towards origin as shown in the figure above.



$z = \frac{1}{\sqrt{x^2+y^2}}$
becomes infinite at the origin.

(3) The requirement that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

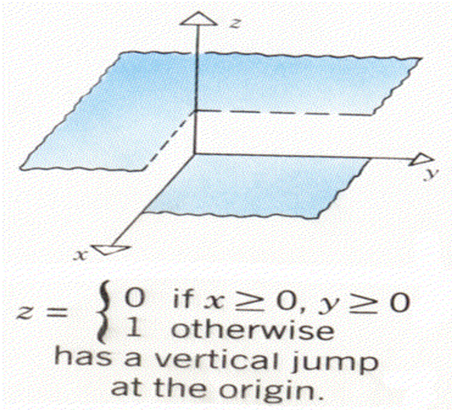
ensures us that the surface z = f(x,y) of the function f(x,y) doesn't have a vertical jump or step above the point (x_0,y_0).

Consider the function of two variables

$$f(x,y) = \begin{cases} 0 & if\ \ x \ge 0 and\ y \ge 0 \\ 1 & otherwise \end{cases}$$

now as we know that the Natural domain of the function is whole the plane. But you should note that the function has one value "0" for all the points in the plane for which both x and y have nonnegative values. And value "1" for all other points in the plane. Consequently the surface

$z = f(x,y) = \begin{cases} 0 & if\ \ x \ge 0 and\ y \ge 0 \\ 1 & otherwise \end{cases}$ has a jump as shown in the figure



Example
Check whether the limit exists or not for the function

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+y^2}$$

Solution:
First we will calculate the Limit of the function along x-axis and we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+0} = 1 \textbf{ (Along x-axis)}$$

Now we will find out the limit of the function along y-axis and we note that the limit is $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{y^2}{y^2+0} = 1$ **(Along y-axis).** Now we will find out the limit of the function along the line y = x and we note that $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$ **(Along y = x)**
It means that limit of the function at (0,0) doesn't exists because it has different values along different paths. Thus the function cannot be continuous at (0,0). And also note that the function id not defined at (0,0) and hence it doesn't satisfy two conditions of the continuity.

Example
Check the continuity of the function at (0,0)

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution:
First we will note that the function is defined on the point where we have to check the Continuity that is the function has value at (0,0). Next we will find out the Limit of the function at (0,0) and in evaluating this limit we use the result $\lim_{x\to0} \frac{\sin x}{x} = 1$ and note that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\mathrm{Sin}(x^2+y^2)}{x^2+y^2}$$
$$=1 = f(0,0)$$

This shows that f is continuous at (0,0)

CONTINUITY OF FUNCTION OF THREE VARIABLES

A function f of three variables is called *continuous at a point* (x_0,y_0,z_0) if

1. $f(x_0,y_0,z_0)$ if defined.
2. $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ exists;
3 $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$.

EXAMPLE

Check the continuity of the function

$$f(x,y,z) = \frac{y+1}{x^2+y^2-1}$$

Solution:

First of all note that the given function is not defined on the cylinder $x^2+y^2-1=0$ .Thus the function is not continuous on the cylinder $x^2+y^2-1=0$ And continuous at all other points of its domain.

RULES FOR CONTINOUS FUNCTIONS

(a) If g and h are continuous functions of one variable, then f(x, y) = g(x)h(y) is a continuous function of x and y

(b) If g is a continuous function of one variable and h is a continuous function of two variables, then their composition f(x, y) = g(h(x,y)) is a continuous function of x and y.

A composition of continuous functions is continuous.
A sum, difference, or product of continuous functions is continuous.
A quotient of continuous function is continuous, expect where the denominator is zero.

EXAMPLE OF PRODUCT OF FUNCTIONS TO BE CONTINUED

In general, any function of the form
$f(x, y) = Ax^m y^n$ (m and n non negative integers)
is continuous because it is the product of the continuous functions Ax^m and y^n.

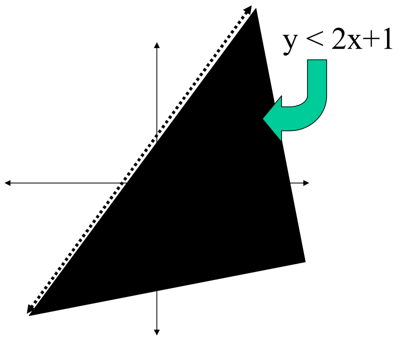
The function $f(x, y) = 3x^2y^5$ is continuous because it is the product of the continuous functions $g(x) = 3x^2$ and $h(y) = y^5$.

CONTINUOUS EVERYWHERE

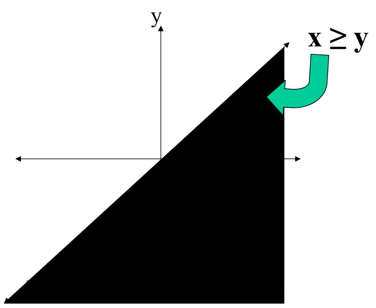
A function f that is continuous at each point of a region R in 2-dimensional space or 3-dimensional space is said to be *continuous on R.* A function that is continuous at every point in 2-dimensional space or 3-dimensional space is called continuous *everywhere* or simply *continuous.*

Example:

(1)f (x,y) = ln(2x – y +1)
The function f is continuous in the whole region where 2x > y-1, y < 2x+1.And its region is shown in figure below.



(2) $f(x, y) = e^{1-xy}$
The function f is continuous in the whole region of xy-plane.
(3) $f(x, y) = \tan^{-1}(y-x)$
The function f is continuous in the whole region of xy- plane.
(4) $f(x, y) = \sqrt{y-x}$
The function is continuous where $x \geq y$



Partial derivative
Let f be a function of x and y. If we hold y constant, say $y = y_0$ and view x as a variable, then $f(x, y_0)$ is a function of x-alone. If this function is differentiable at $x = x_o$, then the value of this derivative is denoted by $f_x(x_o, y_0)$ and is called the Partial derivative of f with respect of x at the point (x_o, y_0).
Similarly, if we hold x constant, say $x = x_0$ and view y as a variable, then $f(x_0, y)$ is a function of y alone. If this function is differentiable at $y = y_0$, then the value of this derivative is denoted by $f_y(x_0, y_0)$ and is called the Partial derivative of f with respect of y at the point (x_0, y_0)

Example

$$f(x, y) = 2x^3y^2 + 2y + 4x$$

Treating y as a constant and differentiating with respect to x, we obtain

$$f_x(x, y) = 6x^2y^2 + 4$$

Treating x as a constant and differentiating with respect to y, we obtain

$$f_y(x, y) = 4x^3y + 2$$

Substituting $x = 1$ and $y = 2$ in these partial-derivative formulas yields.

$$f_x(1, 2) = 6(1)^2(2)^2 + 4 = 28$$
$$f_y(1, 2) = 4(1)^3(2) + 2 = 10$$

Example

$$Z = 4x^2 - 2y + 7x^4y^5$$
$$\frac{\partial z}{\partial x} = 8x + 28x^3y^5$$
$$\frac{\partial z}{\partial y} = -2 + 35x^4y^4$$

Example

$$z = f(x, y) = x^2 sin^2 y$$

Then to find the derivative of f with respect to x we treat y as a constant therefore

$$\frac{\partial z}{\partial x} = f_x = 2x\sin^2 y$$

Then to find the derivative of f with respect to y *we* treat x as a constant therefore

$$\frac{\partial z}{\partial y} = f_y = x^2 2\sin y\cos y$$
$$= x^2 \sin 2y$$

Example

$$z = \ln\left(\frac{x^2 + y^2}{x + y}\right)$$

By using the properties of the ln we can write it as

$$z = \ln(x^2 + y^2) - \ln(x + y)$$
$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + y^2} \cdot 2x - \frac{1}{x + y}$$
$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x^2 + y^2)(x + y)}$$
$$= \frac{x^2 + 2xy - y^2}{(x^2 + y^2)(x + y)}$$

Similarly, (or by symmetry)

$$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x^2 + y^2)(x + y)}$$

Example

$$z = x^4 \sin(xy^3)$$
$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^4 \sin(xy^3)\right]$$
$$= x^4 \frac{\partial}{\partial x}\left[\sin(xy^3)\right] + \sin(xy^3)\frac{\partial}{\partial x}(x^4)$$
$$= x^4 \cos(xy^3)y^3 + \sin(xy^3)4x^3$$
$$\frac{\partial z}{\partial x} = x^4 y^3 \cos(xy^3) + \sin(xy^3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[x^4 \sin(xy^3)\right]$$
$$= x^4 \frac{\partial}{\partial y}\left[\sin(xy^3)\right] + \sin(xy^3)\frac{\partial}{\partial y}(x^4)$$
$$= x^4 \cos(xy^3).3xy^2 + \sin(xy^3).0$$
$$= 3x^5 y^2 \cos(xy^3)$$

Example

$$z = \cos(x^5 y^4)$$
$$\frac{\partial z}{\partial x} = -\sin(x^5 y^4)\frac{\partial}{\partial x}(x^5 y^4)$$
$$= -5x^4 y^4 \sin(x^5 y^4)$$
$$\frac{\partial z}{\partial y} = -\sin(x^5 y^4)\frac{\partial}{\partial y}(x^5 y^4)$$
$$= -4x^5 y^3 \sin(x^5 y^4)$$

Example

$$w = x^2 + 3y^2 + 4z^2 - xyz$$
$$\frac{\partial w}{\partial x} = 2x - yz$$
$$\frac{\partial w}{dy} = 6y - xz$$
$$\frac{dw}{dz} = 8z - xy$$

Lecture No -7 Geometric meaning of partial derivative

Geometric meaning of partial derivative

$$z = f(x, y)$$

Partial derivative of f with respect of x is denoted by

$$\frac{\partial z}{\partial x} \text{ or } f_x \text{ or } \frac{\partial f}{\partial x}$$

Partial derivative of f with respect of y is denoted by

$$\frac{\partial z}{\partial y} \text{ or } f_y \text{ or } \frac{\partial f}{\partial y}$$

Partial Derivatives

Let z = f(x, y) be a function of two variable defined on a certain domain D.
For a given change Δx in x, keeping y as it is, the change Δz in z, is given by

$$\Delta z = f(x + \Delta x, y) - f(x, y)$$

If the ratio

$$\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

approaches to a finite limit as $\Delta x \to 0$, then this limit is called Partial derivative of f with respect of x.
Similarly for a given change Δy in y, keeping x as it is, the change Δz in z, is given by

$$\Delta z = f(x, y + \Delta y) - f(x, y)$$

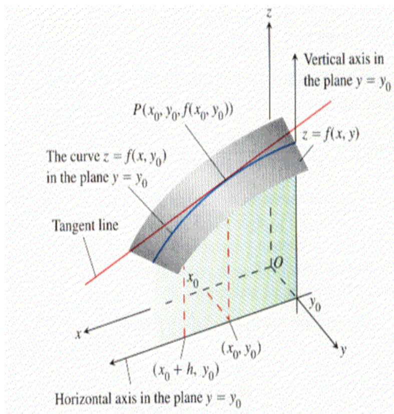
If the ratio

$$\frac{\Delta z}{\Delta y} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

approaches to a finite limit as $\Delta y \to 0$, then this limit is called Partial derivative of f with respect of y.

Geometric Meaning of Partial Derivatives

Suppose $z = f(x, y)$ is a function of two variables. The graph of f is a surface. Let P be a point on the graph with coordinates $(x_o, y_o, f(x_o, y_o))$. If a point starting from P, changes its position on the surface such that y remains constant, then the locus of this point is the curve of intersection of $z = f(x, y)$ and y = constant. On this curve,

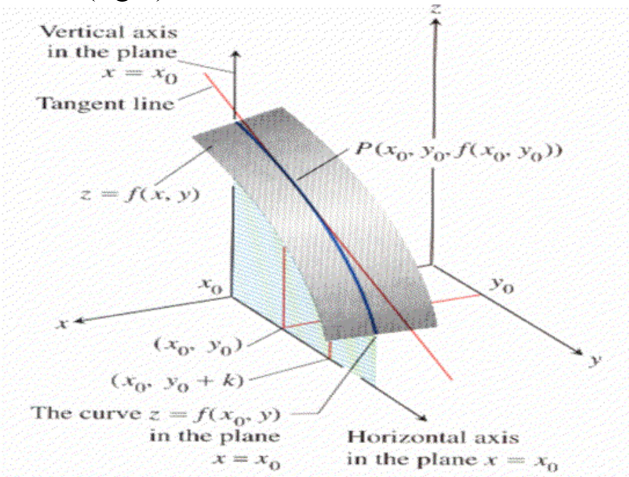


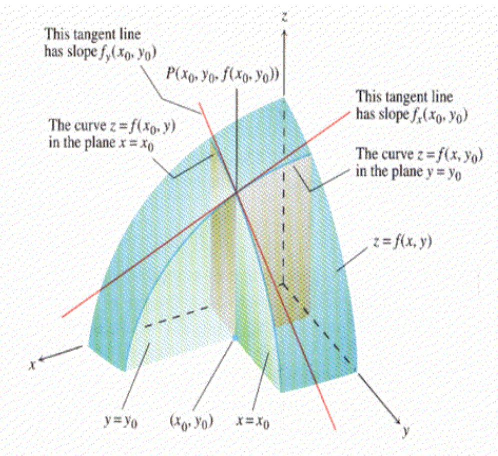
$\frac{\partial z}{\partial x}$ is derivative of $z = f(x, y)$ with respect to x with y constant

Thus $\frac{\partial z}{\partial x}$ = slope of the tangent to this curve at P

Similarly, $\frac{\partial z}{\partial y}$ is the gradient of the tangent at p to the curve of intersection of $z = f(x, y)$ and x = constant.

As shown in the figure below (left) Also together these tangent lines are shown in figure below (right).





Partial Derivatives of Higher Orders

The partial derivatives f_x and f_y of a function f of two variables x and y, being functions of x and y, may possess derivatives. In such cases, the **second order** partial derivatives are defined as below.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx} = f_x^2$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy} = f_y^2$$

Thus, there are four second order partial derivatives for a function $z = f(x, y)$. The partial derivatives f_{xy} and f_{yx} are called **mixed second partials** and **are not equal** in general. Partial derivatives of order more than two can be defined in a similar manner.

Example

$$z = \arcsin\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{1}{y} = \frac{1}{\sqrt{y^2 - x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{-x}{y^2} = \frac{-x}{y\sqrt{y^2 - x^2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{-1}{2}(y^2 - x^2)^{-3/2} 2y = \frac{-y}{(y^2 - x^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{-1}{y\sqrt{y^2 - x^2}} - \frac{x}{y}\left[\frac{x}{(y^2 - x^2)^{3/2}}\right]$$

$$= \frac{-y^2 + x^2 - x^2}{y(y^2 - x^2)^{3/2}} = \frac{-y}{(y^2 - x^2)^{3/2}}$$

Hence

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$

Example

$$f(x, y) = x\cos y + ye^x$$

$$\frac{\partial f}{\partial x} = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$$

$$f(x, y) = x\cos y + y e^x$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y$$

Laplace's Equation

For a function ***w = f(x,y,z)***

The equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \quad \frac{\partial^2 f}{\partial z^2} = 0$$

is called **Laplace's equation** .

Example

$$f(x, y) = e^x \sin y + e^y \cos x,$$

$$\frac{\partial f}{\partial x} = e^x \sin y - e^y \sin x$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \sin y - e^y \cos x$$

$$\frac{\partial f}{\partial y} = e^x \cos y + e^y \cos x$$

$$\frac{\partial^2 f}{\partial y^2} = -e^x \sin y + e^y \cos x$$

Adding both partial second order derivative, we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x \sin y - e^y \cos x - e^x \sin y + e^y \cos x = 0$$

Euler's theorem
The mixed derivative theorem

If f(x,y) and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a , b) = f_{yx} (a , b)$$

Advantage of Euler's theorem

$$w = xy + \frac{e^y}{y^2 + 1}$$

The symbol $\partial^2 w / \partial x \, \partial y$ tell us to differentiate first with respect to y and then with respect to x. However, if we postpone the differentiation with respect to y and differentiate first with respect to x, we get the answer more quickly.

$$\frac{\partial w}{\partial x} = y \qquad \frac{\partial^2 w}{\partial y \partial x} = 1$$

Overview of lecture# 7
Chapter # 16 Partial derivatives
Page # 790 Article # 16.3

Lecture No- 8 More About Euler Theorem Chain Rule

In general, the order of differentiation in an nth order partial derivatice can be change without affecting the final result whenever the function and all its partial derivatives of order $\leq$ n are continuous. For example, if f and its partial order derivatives of the first, second, and third orders are continuous on an open set, then at each point of the set,

$$f_{xyy} = f_{yxy} = f_{yyx}$$

or in another notation.

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial^3 f}{\partial x \partial y^2}$$

Order of differentiation

For a function

$$f(x,y) = y^2 x^4 e^x + 2$$

If we are interested to find $\frac{\partial^5 f}{\partial y^3 \partial x^2}$, that is, differentiating in the order firstly w.r.t. x and then w.r.t. y, calculation will involve many steps making the job difficult. But if we differentiate this function with respect to y, firstly and then with respect to x secondly then the value of this fifth order derivative can be calculated in a few steps.

$$\frac{\partial^5 f}{\partial x^2 \partial y^3} = 0$$

EXAMPLE

$$f(x,y) = \frac{x+y}{x-y}$$

$$f_x(x,y) = \frac{(x-y)\frac{\partial}{\partial x}(x+y) - (x+y)\frac{\partial}{\partial x}(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)(1) - (x+y)(1)}{(x-y)^2}$$

$$= \frac{-2y}{(x-y)^2}$$

$$f_y(x,y) = \frac{(x-y)\frac{\partial}{\partial y}(x+y) - (x+y)\frac{\partial}{\partial y}(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)(1) - (x+y)(-1)}{(x-y)^2}$$

$$= \frac{2x}{(x-y)^2}$$

EXAMPLE

$$f(x,y) = x^3 e^{-y} + y^3 \sec\sqrt{x}$$

$$f_x(x,y) = 3x^2 e^{-y} + y^3 \sec\sqrt{x}\tan\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

$$f_y(x,y) = -x^3 e^{-y} + 3y^2 \sec\sqrt{x}$$

EXAMPLE

$$f(x,y) = x^2 y e^{xy}$$

$$f_x(x,y) = 2xye^{xy} + x^2 y^2 e^{xy}$$
$$= xye^{xy}(2 + xy)$$
$$f_x(1,1) = (1)(1)e^{(1)(1)}[2 + (1)(1)]$$
$$= 3e$$

$$f(x,y) = x^2 y e^{xy}$$

$$f_y(x,y) = x^2 e^{xy} + x^3 y e^{xy}$$
$$= x^2 e^{xy}(1 + xy)$$
$$f_y(1,1) = (1)(1)e^{(1)(1)}[1 + (1)(1)]$$
$$= 2e$$

Example

$$f(x,y) = x^2 \cos(xy)$$
$$f_x(x,y) = 2x\cos(xy) - x^2 y \sin(xy)$$
$$f_x(1/2, \pi) = 2(1/2)\cos(\pi/2) - (1/2)^2(\pi)\sin(\pi/2)$$
$$= -\pi/4$$

$$f_y(x,y) = -x^3 \sin(xy)$$
$$f_y(1/2, \pi) = -(1/2)^3 \sin(\pi/2)$$
$$= -1/8$$

EXAMPLE

$$w = (4x - 3y + 2z)^5$$

$$\frac{\partial w}{\partial x} = 20(4x - 3y + 2z)^4$$

$$\frac{\partial^2 w}{\partial y \partial x} = -24 \ \ (4x - 3y + 2z)^3$$

$$\frac{\partial^3 w}{\partial z \partial y \partial x} = -1440(4x-3y+2z)^2$$

$$\frac{\partial^4 w}{\partial z^2 \partial y \partial x} = -576 \ \ (4x - 3y + 2z)$$

Chain Rule in function of One variable

Given that w= f(x) and x = g(t), we find $\frac{dw}{dt}$ *as follows:*

From w = f(x), we get $\frac{dw}{dx}$

From x = g(t), we get $\frac{dx}{dt}$

Then

$$\frac{dw}{dt} = \frac{dw}{dx} \ \frac{dx}{dt}$$

Example

w = x + 4, x = Sint

By Substitution

w = Sint + 4

$$\frac{dw}{dt} = \text{Cost}$$

$$w = x + 4 \quad \Rightarrow \frac{dw}{dx} = 1$$

$$x = \text{Sint} \quad \Rightarrow \frac{dx}{dt} = \text{Cost}$$

By Chain Rule

$$\frac{dw}{dt} = \frac{dw}{dt} \times \frac{dx}{dt} = 1 \ . \times \text{Cost} = \text{Cost}$$

Chain rule in function of one variable

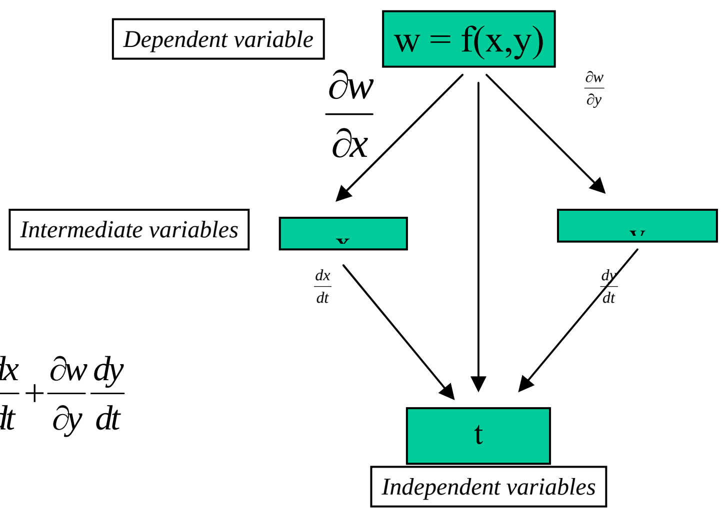
y is a function of u, u is a function of v
v is a function of w, w is a function of z
z is a function of x. Ultimately y is function of x

so we can talk about $\dfrac{dy}{dx}$

and by chain rule it is given by

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dz}\frac{dz}{dx}$$

w = f(x,y), x = g(t), y = f(t)



$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}$$

<u>EXAMPLE BY SUBSTITUTION</u>

$$w = xy$$
$$x = \cos t,\ y = \sin t$$
$$w = \cos t \sin t$$
$$= \frac{1}{2}\ 2 \sin t \cot$$
$$= \frac{1}{2}\ \sin 2t$$
$$\frac{dw}{dt} = \frac{1}{2}\ \cos 2t.2$$
$$= \cos 2t$$

EXAMPLE

$$w = xy, x = \cos t, \text{and } y = \sin t$$

$$\frac{\partial w}{\partial y} = x \qquad \frac{\partial w}{\partial x} = y$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t,$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t$$

EXAMPLE

$$\mathbf{z = 3x^2 y^3}$$
$$\mathbf{x = t^4 , \ y = t^2}$$

$$\frac{\partial z}{\partial x} = 6xy^3, \qquad \frac{\partial z}{\partial y} = 9x^2y^2$$

$$\frac{dx}{dt} = 4t^3, \qquad \frac{dy}{dt} = 2t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (6xy^3)(4t^3) + 9x^2y^2(2t)$$
$$= 6(t^4)(t^6)(4t^3) + 9(t^8)(t^4)(2t)$$
$$= 24t^{13} + 18t^{13} = 42t^{13}$$

EXAMPLE

$$\mathbf{z = \sqrt{1 + x - 2xy^4}}$$
$$\mathbf{x = \ln}$$
$$\mathbf{y = t}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{1 + x - 2xy^4}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{1 + x - 2xy^4}} .(-8xy^3) = \frac{-4xy^3}{\sqrt{1 + x - 2xy^4}}$$

$$\frac{dx}{dt} = \frac{1}{t}, \qquad \frac{d}{dt} =$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1-2y^4}{2\sqrt{1+x-2xy^4}\;t}\;\frac{1}{t} - \frac{4xy^3}{\sqrt{1+x-2xy^4}}\cdot 1$$

$$= \frac{1}{\sqrt{1+x-2xy^4}}\left[\frac{1-}{2t} - 4xy^3\right]$$

$$= \frac{1}{\sqrt{1+\text{lint}-2(\ln t)\,t^4}}\left[\frac{1-2t^4}{2t} - 4(\ln t)\,t^3\right]$$

$$= \frac{1}{\sqrt{1+\ln t-2t^4\ln t}}\left[\frac{1}{2t} - t^3 - 4t^3\ln t\right]$$

EXAMPLE

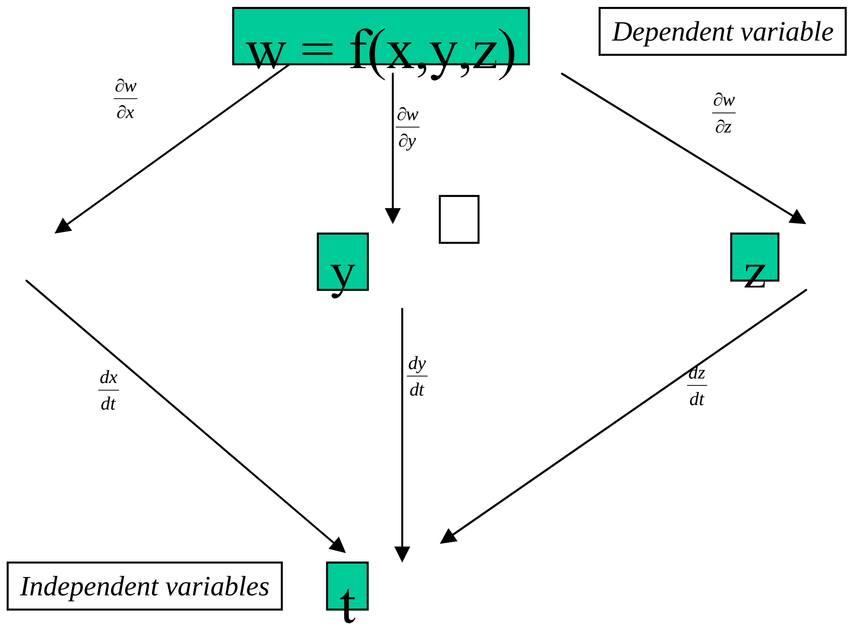
$$\mathbf{z = \ln(2x^2 + y)}$$

$$\mathbf{x = \sqrt{t}\,,\qquad y = t^{2/3}}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2x^2+y}\cdot 4x = \frac{4}{2x^2+y}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2x^2+y},\quad \frac{dx}{dt} = \frac{1}{2}\frac{1}{\sqrt{t}},\qquad \frac{dy}{dt} = \frac{2}{3}t^{-1/3}$$

$w = f(x,y,z),\ x = g(t)\ ,y = f(t),\ z = h(t)$



Overview of Lecture#8

Chapter # 16
Topic # 16.4
Page # 799 Book Calculus By Haward Anton

Lecture No - 9 Examples

First of all we revise the example which we did in our 8th lecture.

Consider w = f(x,y,z) Where

x = g(t), y = f(t), z = h(t)

Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example:

$$\mathbf{w = x^2 + y + z + 4}$$

$$\mathbf{x = e^t, \quad y = \cos t, \quad z = t + 4}$$

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 1, \quad \frac{\partial w}{\partial z} = 1$$

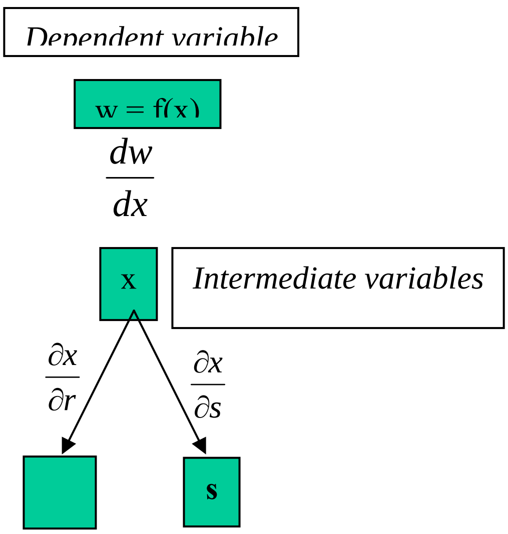
$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = -\text{Sint}, \quad \frac{dz}{dt} = 1$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt} \\ &= (2x)(e^t) + (1).(-\text{Sint}) + (1)(1) \\ &= 2(e^t)(e^t) - \text{Sint} + 1 \\ &= 2e^{2t} - \text{Sint} + 1\end{aligned}$$

Consider

w = f(x), where x = g(r, s). Now it is clear from the figure that "x" is intermediate variable and we can write.

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$



Example:

$$\mathbf{w = Sin\ x + x^2, \quad x = 3r + 4s}$$

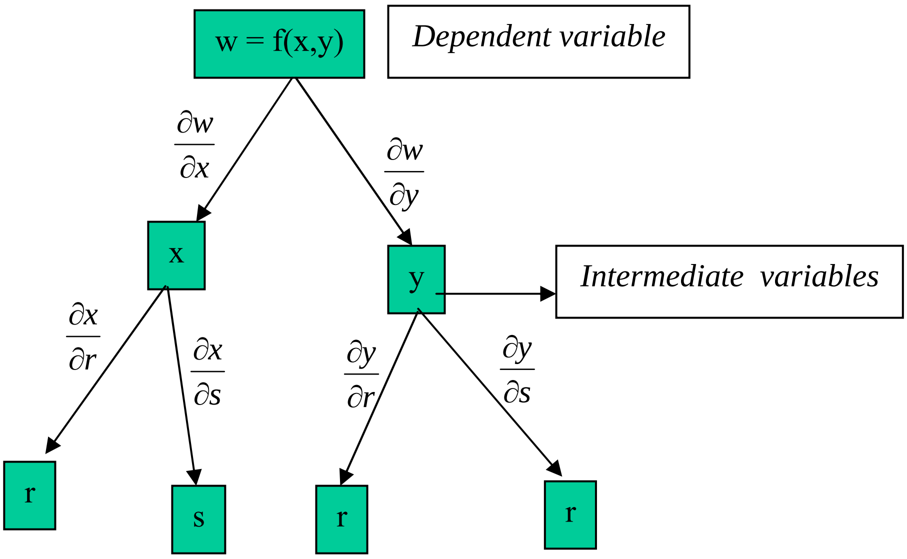
$$\frac{dw}{dx} = \text{Cosx} + 2x$$

$$\frac{\partial x}{\partial r} = 3 \qquad \frac{\partial x}{\partial s} = 4$$

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{dw}{dx}\cdot\frac{\partial x}{\partial r} \\ &= (\text{Cosx} + 2x) . 3 \\ &= 3\,\text{Cos}(3r+4s) + 6(3r + 4s) \\ &= 3\,\text{Cos}(3r + 4s) + 18r + 24s\end{aligned}$$

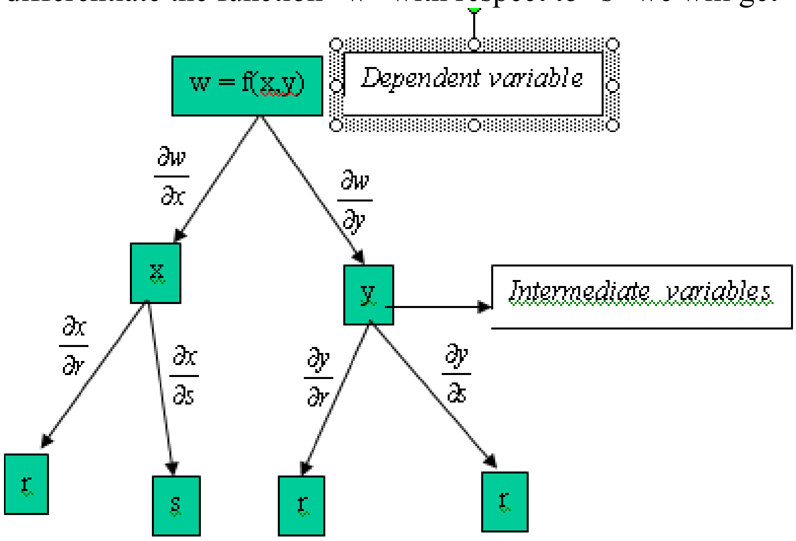
$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{dw}{dx}\cdot\frac{\partial x}{\partial s} \\ &= (\text{Cosx} + 2x) . 4 \\ &= 4\,\text{Cosx} + 8x \\ &= 4\,\text{Cos}(3r + 4s) + 8(3r + 4s) \\ &= 4\,\text{Cos}(3r + 4s) + 24r + 32s\end{aligned}$$

Consider the function $w = f(x,y)$, *Where* $x = g(r, s), y = h(r, s)$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

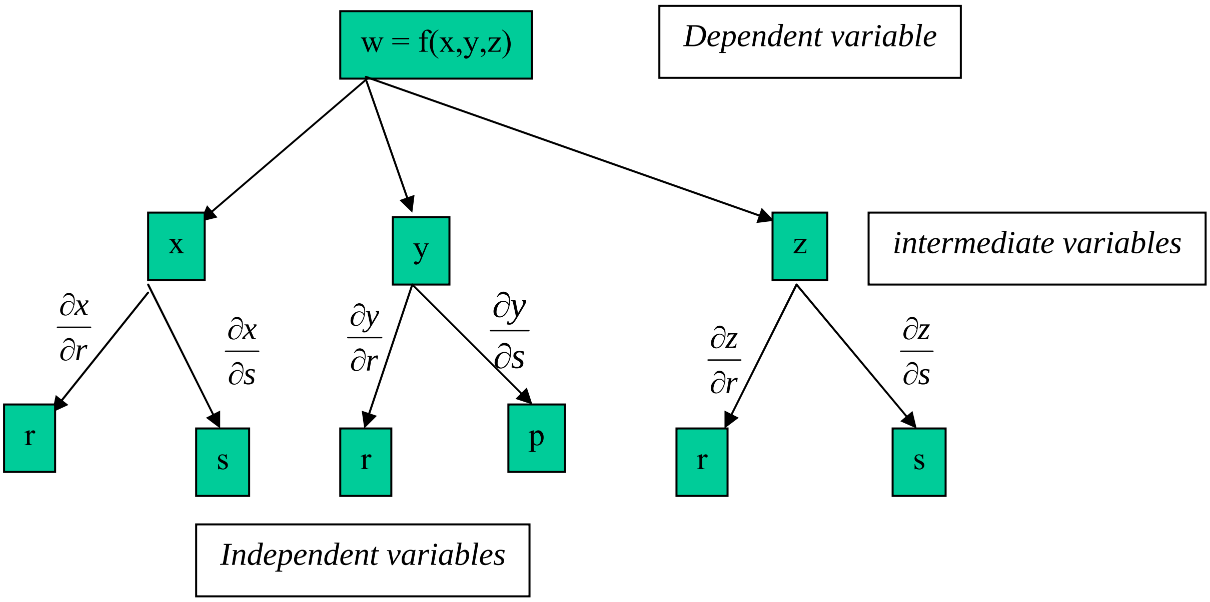
Similarly if you differentiate the function "w" with respect to "s" we will get



And we have

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Consider the function *w = f(x,y,z), Where x = g(r, s), y = h(r,s), z = k(r, s)*



Thus we have

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Similarly if we differentiate with respect to "s" then we have,

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example:

Consider the function $w = x + 2y + z^2,\ x = \frac{r}{s}, y = r^2 + \ln s,\ z = 2r$

First we will calculate

$$\frac{\partial w}{\partial x}=1 \quad \frac{\partial w}{\partial y}=2 \quad \frac{\partial w}{\partial z}=2z \quad \frac{\partial x}{\partial r}=\frac{1}{s} \quad \frac{\partial y}{\partial r}=2r \quad \frac{\partial z}{\partial r}=2$$

Now as we know that $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$ By putting the values from above we get

$$\frac{\delta w}{\delta r}=(1)\left(\frac{1}{s}\right)+(2)(2r)+(2z)(2)$$

$$=\frac{1}{s}+4r+(4r)(2) \quad =\frac{1}{s}+12r$$

Now

$$\frac{\partial x}{\partial s}=-\frac{r}{s^2} \quad \frac{\partial y}{\partial s}=\frac{1}{s} \quad \frac{\partial z}{\partial s}=0$$

So we can calculate $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

$$=(1)\left(-\frac{r}{s^2}\right)+(2)\left(\frac{1}{s}\right)+(2z)(0)$$

$$=\frac{2}{s}-\frac{r}{s^2}$$

Remembering the different Forms of the chain rule:
The best thing to do is to draw appropriate tree diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom. To find the derivative of dependent variable with respect to the selected independent variable, start at the dependent variable and read down each branch of the tree to the independent variable, calculating and multiplying the derivatives along the branch. Then add the products you found for the different branches.

The Chain Rule for Functions of Many Variables

Suppose $\omega = f(x, y, \ldots, \upsilon)$ is a differentiable function of the variables $x, y, \ldots, \upsilon$ (a finite set) and the $x, y, \ldots, \upsilon$ are differentiable functions of p, q, , t (another finite set). Then ω is a differentiable function of the variables p through t and the partial derivatives of ω with respect to these variables are given by equations of the form

$$\frac{\partial\omega}{\partial p} = \frac{\partial\omega}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial\omega}{\partial y}\frac{\partial y}{\partial p} + \ldots\ldots + \frac{\partial\omega}{\partial \upsilon}\frac{\partial \upsilon}{\partial p}.$$

The other equations are obtained by replacing p by q, …, t, one at a time. One way to remember last equation is to think of the right-hand side as the dot product of two vectors with components.

$$\underbrace{\left(\frac{\partial\omega}{\partial x}, \frac{\partial\omega}{\partial y} \ldots\ldots \frac{\partial\omega}{\partial \upsilon}\right)}_{\text{Derivatives of } \omega \text{ with respect to the intermedaite variables}} \quad \text{and} \quad \underbrace{\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p} \ldots\ldots \frac{\partial \upsilon}{\partial p}\right)}_{\text{Derivatives of the intermedaite variables with respect to the selected independent variable}}$$

Example:

$$w = \ln(e^r + e^s + e^t + e^u)$$

Taking "ln" of both sides of the given equation we get

$$e^w = e^r + e^s + e^t + e^u$$

Now Taking partial derivative with respect to "r, s , u , and t" we get

$$e^w w_r = e^r \Rightarrow w_r = e^{r-w} \quad , \quad e^w w_s = e^s \Rightarrow w_s = e^{s-w} \quad , \quad e^w w_u = e^u \Rightarrow w_u = e^{u-w} \text{ and}$$

$$e^w w_t = e^t \Rightarrow w_t = e^{t-w}$$

Now since we have $w_r = e^{r-w}$ Now Differentiate it partially w.r.t. "s"

$$w_{rs} = e^{r-w}(-w_s)$$

$$= -e^{r-w}e^{s-w}$$ (Here we use the value of w_s)

$$w_{rs} = -e^{r+s-2w}$$

Now differentiate it partially w.r.t. "t" and using the value of w_t we get,

$$w_{rst} = -e^{r+s-2w}(-2w_t)$$
$$= 2e^{r+s-2w}e^{t-w}$$
$$w_{rst} = 2e^{r+s+t-3w}$$

Now differentiate it partially w.r.t. "u" we get,

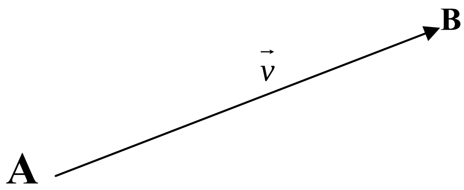
$$w_{rstu} = 2e^{r+s-3w}(-3w_u)$$ and by putting the value of w_u, we get,

$$w_{rstu} = -6e^{r+s+t-3w}(e^{u-w})$$

$$w_{rstu} = -6e^{r+s+t+u-4w}$$

Lecture No -10 Introduction to vectors

Some of things we measure are determined by their magnitude. But some times we need magnitude as well as direction to describe the quantities.
For Example, To describe a force, We need direction in which that force is acting (Direction) as well as how large it is (Magnitude).
Other Example is the body's Velocity; we have to know where the body is headed as well as how fast it is.
Quantities that have direction as well as magnitude are usually represented by arrows that point the direction of the action and whose lengths give magnitude of the action in term of a suitably chosen unit.
A vector in the plane is a directed line segment.



$$\vec{v} = \overrightarrow{AB}$$

Vectors are usually described by the single bold face roman letters or letter with an arrow. The vector defined by the directed line segment from point A to point B is written as $\overrightarrow{AB}$.

Magnitude or Length Of a Vector :

Magnitude of the vector $\vec{v}$ is denoted by

$|\vec{v}| = |\overrightarrow{\mathbf{AB}}|$ is the length of the line segment AB

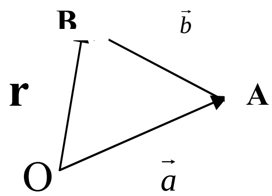
Unit vector

Any Vector whose Magnitude or length is 1 is a unit vector.

Unit vector in the direction of vector $\vec{v}$ is denoted by $\hat{v}$.
and is given by

$$\hat{v} = \frac{\vec{v}}{|v|}$$

Addition Of Vectors



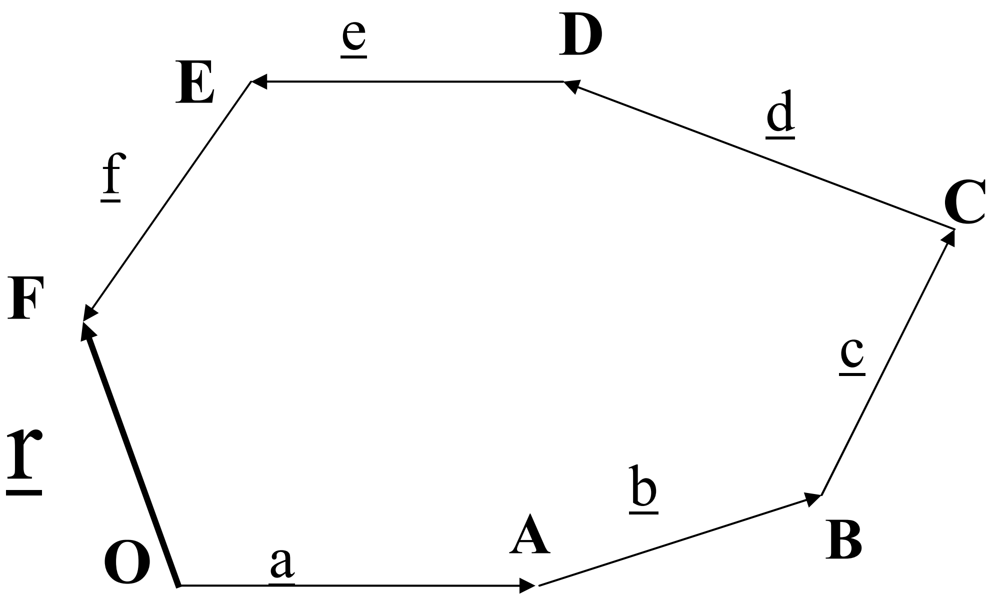
This diagram shows three vectors, in two vectors one vector $\overrightarrow{OA}$ is connected with tail of vector $\overrightarrow{AB}$. The tail of third vector $\overrightarrow{OB}$ is connected with the tail of OA and head is connected with the head of vector $\overrightarrow{AB}$.This third vector is called Resultant vector.

The resultant vector can be written as

$$\underline{r} = \underline{a} + \underline{b}$$

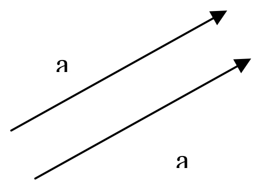
Similarly

$$\underline{r} = \underline{a} + \underline{b} + \underline{c} + \underline{d} + \underline{e} + \underline{f}$$



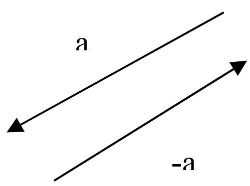
Equal vectors

Two vectors are equal or same vectors if they have same magnitude and direction.



Opposite vectors

Two vectors are opposite vectors if they have same magnitude and opposite direction.

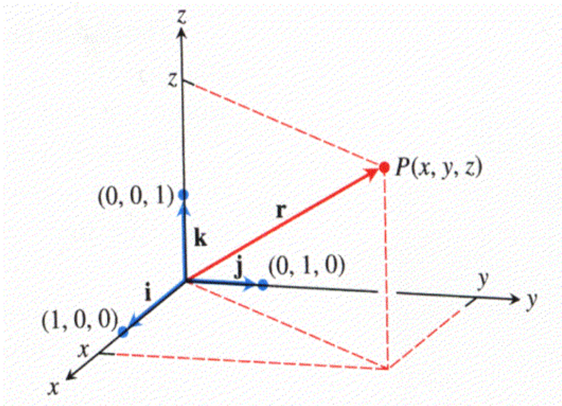


Parallel vectors

Two vector are parallel if one vector is scalar multiple of the other.

$$\underline{b} = \lambda \underline{a}$$

where λ is a non zero scalar.



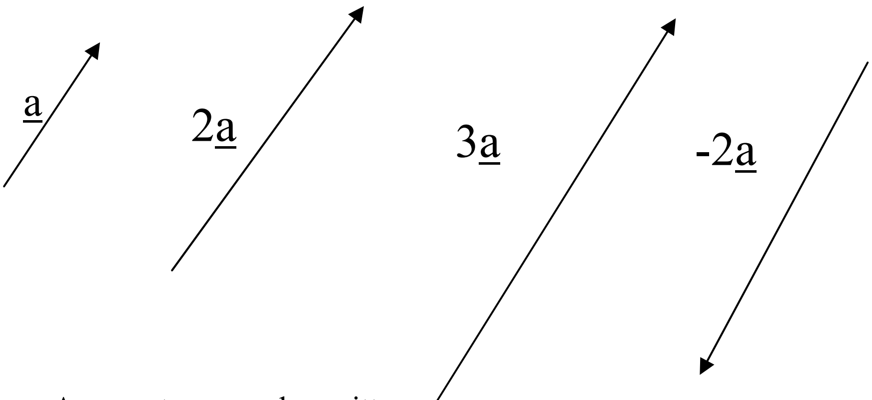
$\underline{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$

Addition and subtraction of two vectors in rectangular component:

$$
\begin{aligned}
\text{Let}\quad \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\
\text{and}\quad \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \\
\mathbf{a} + \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \\
\mathbf{a} - \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) - (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}
\end{aligned}
$$

I th component of first vector is added (subtracted) to the ith component of second vector, jth component of first vector is added (subtracted) to the jth component of second vector, similarly kth component of first vector is added (subtracted) to the kth component of second vector,

Multiplication of a vector by a scalar



Any vector **a** can be written as

$$\underline{a} = |\underline{a}|\,\hat{a}$$

Scalar product

Scalar product (dot product) (“a dot b”) of vector a and b is the number

$$a.b = |a|\,|b| \cos\theta.$$

where θ is the angle between a and b.

In word, a.b is the length of a times the length of b times the cosine of the angle between a and b.

Remark:-

$$\mathbf{a.b = b.a}$$

This is known as commutative law.

Some Results of Scalar Product

$$a.b = |a|\,|b| \cos\theta.$$

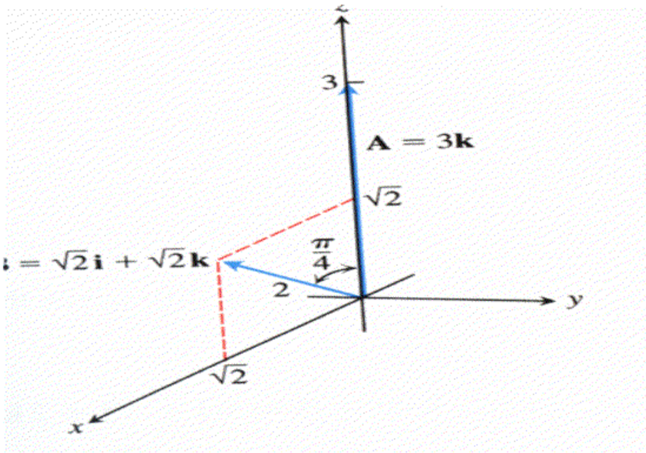
1. $a \perp b$ If
   This means that if a is perpendicular to b.
   Then $a.b=0$
   Also
   $i.j= 0 = j.i$
   $j.k= 0 =k.j$
   $k.i= 0 =i.k$

2. If $a \parallel b$
   That means a is parallel to b.
   Then
   $$a.b = |a|\,|b|$$
   if we replace b by a then
   $$\mathbf{a.a} = |\mathbf{a}|\,|\mathbf{a}|$$
   $$\mathbf{a.a} = |\mathbf{a}|^2$$
   $$\sqrt{|\mathbf{a}|} = \mathbf{a.a}$$
   so $\mathbf{i.i=j.j=k.k=1}$

Example

If $\mathbf{a} = 3\mathbf{k}$ and $\mathbf{b} = \sqrt{2}\mathbf{i} + \sqrt{2}\,\mathbf{k}$,
then

$$\mathbf{a}\ . \quad = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$
$$= (3)\,(2) \cos\frac{\pi}{4}$$
$$= 6\ .\ \frac{\sqrt{2}}{2} = 3\sqrt{2}\ .$$



EXPRESSION FOR a.b IN COMPONENT FORM

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\begin{aligned}
\mathbf{a.b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \,.\, (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \,.\, (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \,.\, (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\quad + a_3\mathbf{k} \,.\, (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1b_1\mathbf{i} \,.\, \mathbf{i} + a_1b_2\mathbf{i.j} + a_1b_3\mathbf{i} \,.\, \mathbf{k} + a_2b\mathbf{j} \,.\, \mathbf{i} + a_2b_2\mathbf{j} \,.\, \mathbf{j} \\
&\quad + a_2b_3\mathbf{j} \,.\, \mathbf{k} + a_3b_1\mathbf{k} \,.\, \mathbf{i} + + a_3b_2\mathbf{k} \,.\, \mathbf{j} + a_3b_3\mathbf{k} \,.\, \mathbf{k} \\
&= a_1b_1(1) + a_1b_2(0) + a_1b_3\,(0) + a_2b_1(0) + a_2b_2\,(1) \\
&\quad + a_2b_3(0) + a_3b_1(0) + \;+ a_3b_2(0) + a_3b_3(1) \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}$$

In dot product ith component of vector a will multiply with ith component of vector b , jth component of vector a will multiply with jth component of vector b and kth component of vector a will multiply with kth component of vector b

Angle Between two vectors

The angle between two nonzero vectors **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{\mathbf{a.\,b}}{|\mathbf{a}\,||\mathbf{b}\,|}\right)$$

Since the values of the arc lie in $[0, \pi]$, above equation automatically gives the angle made by **a** and **b**.

Example

a = i − 2j − 2k and b = 6i + 3j + 2 k.

$$\begin{aligned}
\mathbf{a.b} &= (1)(6) + (-2)(3) + (-2)\,(2) \\
&= 6 - 6 - 4 = -4
\end{aligned}$$

$$|\mathbf{a}\,| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{b}\,| = \sqrt{(6)^2 + (\;\;3)^2 + (\;\;2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{a.b}}{|\mathbf{a}\,||\mathbf{b}\,|}\right)$$

$$= \cos^{-1}\left(\frac{-4}{(3)\,(7)}\right) = \cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76 \text{ rad}$$

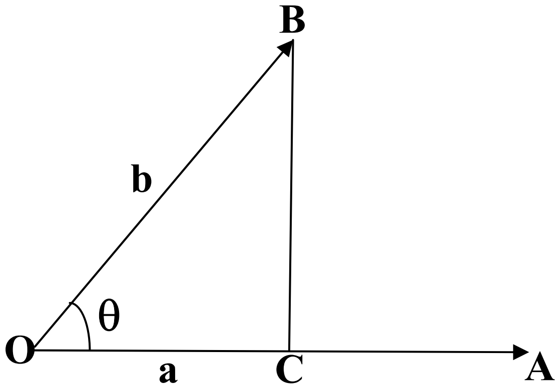
Perpendicular (Orthogonal) Vectors

a and b are perpendicular if and only if a.b = 0.

This has two parts If “a” and “b” are per perpendicular then a.b=0. And if a.b=0 “a” and “b” will be perpendicular.

Vector Projectio

Consider the Projection of a vector b on a vector a making an angle θ with each other



From right angle triangle OCB

COS θ =Base / hypotenuse

$$\text{COS } \theta = \frac{|\vec{OC}|}{|\mathbf{b}|}$$

$$|\vec{OC}| = |\mathbf{b}| \text{ Cos}\theta$$

$$= \frac{|\mathbf{b}|\ |\mathbf{a}| \text{ Cos}\theta}{|\mathbf{a}|}$$

$$= \frac{\mathbf{b.a}}{|\mathbf{a}|} = \mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|}$$

$$= \mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|}$$

The number $|\mathbf{b}| \cos\theta$ is called the **scalar component of B in the direction of a** .

since $|\mathbf{b}| \cos\theta = \text{B}. \frac{\mathbf{a}}{|\mathbf{a}|}$,

we can find the scalar component by "dotting" **b** with the direction of **a**

Example

Vector Projection of $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
onto $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is

$$\text{proj}_{\mathbf{a}}\, \text{b} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a}$$

$$= \frac{6-6-4}{1+4+4}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

$$= -\frac{4}{9}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

$$= \frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.$$

he scalar component of **b** in the direction of **a**

$$|\mathbf{b}|\cos\theta = \mathbf{b}\cdot\frac{\mathbf{a}}{|\mathbf{a}|} = (6\mathbf{i}+3\mathbf{j}+2\mathbf{k}).\left(\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}\right)$$

$$= 2-2-\frac{4}{3} = -\frac{4}{3}.$$

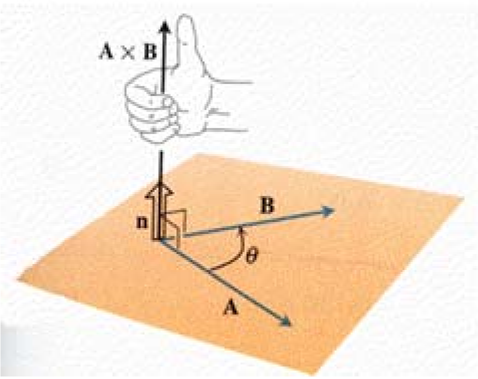
The Cross Product of Two Vectors in Space

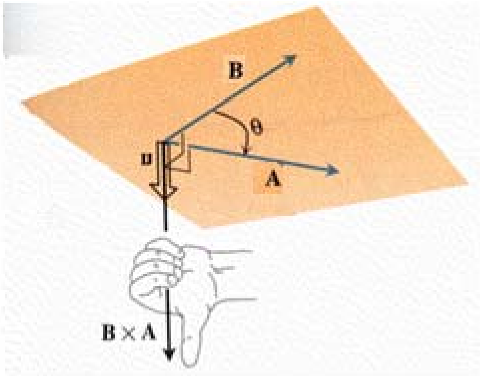
Consider two nonzero vectors **a** and **b** in space. The **vector product** **a** × **b** (" **a** cross **b** ") to be the vector

$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\,\mathbf{n}$

where **n** is a vector determined by right hand rule.

Right-hand rule





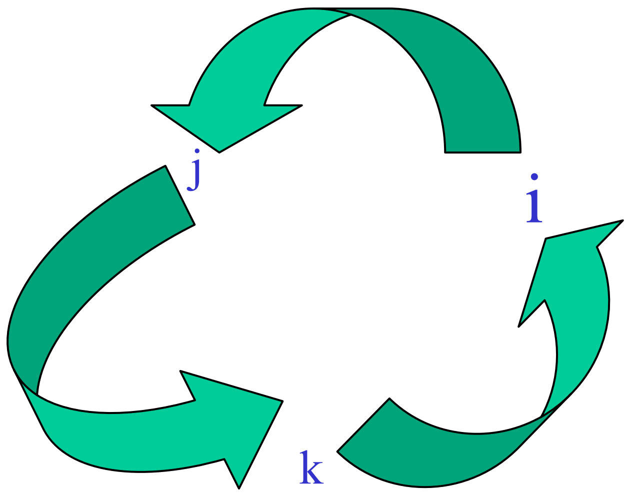
We start with two nonzero nonparallel vectors **A** and **B** .We select a unit vector **n** Perpendicular to the plane by the **right handed rule.** This means we choose n to be the unit vector that points the way your right thumb points when your fingers curl through the angle 0 from **A** to **B** .
The vector A*B is orthogonal to both **A** and **B.**

Some Results of Cross Product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \text{Sin}\ \theta\ \hat{\mathbf{n}}$
If $\mathbf{a} \parallel \mathbf{b}$
then $\mathbf{a} \times \mathbf{b} = 0$
So $\mathbf{a} \times \mathbf{a} = 0$
$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$
If $\mathbf{a} \perp \mathbf{b}$
then $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \hat{\mathbf{n}}$

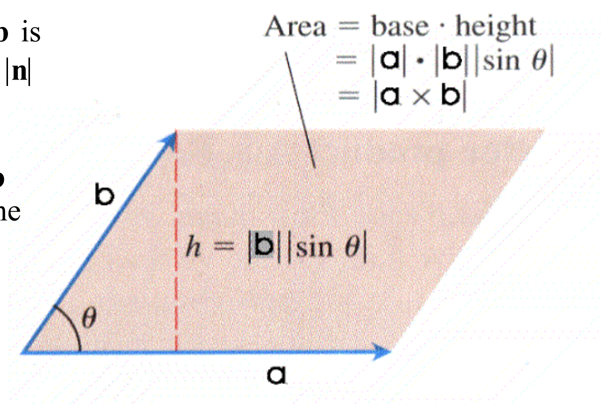
$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Note that this product is not commutative.



The Area of a Parallelogram

Because n is a unit
the magnitude of $\mathbf{a} \times \mathbf{b}$ is
$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\ |\mathbf{b}|\ |\sin\theta|\ |\mathbf{n}|$
$= |A|\ |\mathbf{b}|\ \sin\theta$
This is the area of the
determined by $\mathbf{a}$ and $\mathbf{b}$
$|\mathbf{a}|$ being the base of the
and $|\mathbf{b}|\ |\sin\theta|$ the



$\mathbf{a} \times \mathbf{b}$ from the components of a and b

Suppose

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$
$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} \\ &\quad + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k} \\ &\quad + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} \\ &\quad + (a_1b_2 - a_2b_1)\mathbf{k}.\end{aligned}$$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example:

Let

$\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ **. Then**

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$a \times b = i(1-3) - j(2+4) + k(6+4)$$

$a \times b = -2i - 6j + 10k$ is the required cross product of **a** and **b.**

<u>Over view of Lecture # 10</u>

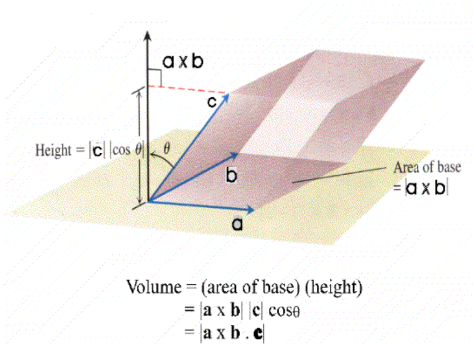
Chapter# 14

Article # 14.3, 14.4

Page # 679

Lecture No -11 The Triple Scalar or Box Product

The product $(\mathbf{a}\times\mathbf{b})\,.\,\mathbf{c}$ is called the **triple scalar product** of **a**, **b**, and **c** (in that order).
As $|\,(\mathbf{a}\times\mathbf{b}).\mathbf{c}\,| = |\mathbf{a}\times\mathbf{b}|\,|\mathbf{c}|\,|\cos\theta|$
the absolute value of the product is the volume of the parallelepiped (parallelogram-sided box) determin-ed by a,b and c



By treating the planes of b and c and of c and a as he base planes of the parallelepiped determined by a, b and c
we see that

$$(a * b).c = (b\times c).a = (c\times a).b$$

Since the dot product is commutative, $(a \times b).c = a.(b\times c)$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$
$$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

$$\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c}) = \mathbf{a}.\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \mathbf{a}\,.\left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\mathbf{k}\right]$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

<u>Example</u>

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},\quad \mathbf{b} = -2\mathbf{i} + 3\mathbf{k},\quad \mathbf{c} = 7\mathbf{j} - 4\mathbf{k}.$$

$$\mathbf{a}\,.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}$$

$$= -21 - 16 + 14$$
$$= -23$$

The volume is

$$|\mathbf{a}\,.(\mathbf{b}\times\mathbf{c})| = 23.$$

When we solve **a.(b×c)** then answer is -23 . if we get negative value then Absolute value make it positive and also volume is always positive.

Gradient of a Scalar Function

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

∇ is called "del operator"

Gradient ϕ is a vector operator defined as

gradient ϕ is a vector operator defined as

$$\text{grad}\,\phi = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]\phi$$

$$= \nabla\phi,$$

$$\text{grand}\,\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \nabla\phi,$$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

∇ is called "del operator"

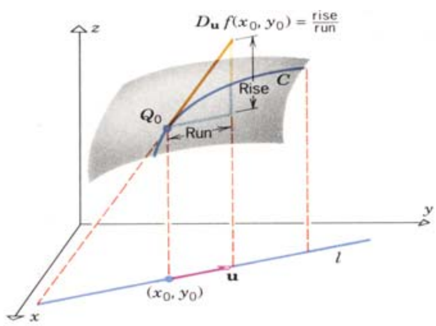
∇ "del operator" is a vector quantity. Grad means gradient. Gradient is also vector quantity. $\nabla\phi$ is vector and ϕ is scalar quantity, Every component of $\nabla\phi$ will operate with the .

Directional Derivative

If $f(x,y)$ is differentiable at (x_0,y_0), and if $\mathbf{u} = (u_1, u_2)$ is a unit vector, then the ***directional derivative*** of f at (x_0, y_0) in the direction of $\mathbf{u}$ is defined by

$$D_{\mathbf{u}} f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$$

It should be kept in mind that there are infinitely many directional derivatives of $z = f(x,y)$ at a point (x_0,y_0), one for each possible choice of the direction vector $\mathbf{u}$



Remarks (Geometrical interpretation)

The directional derivative $D_{\mathbf{u}}f(x_0,y_0)$ can be interpreted algebraically as the instantaneous rate of change in the direction of $\mathbf{u}$ at (x_0,y_0) of $z=f(x,y)$ with respect to the distance parameter s described above, or geometrically as the rise over the run of the tangent line to the curve C at the point Q_0

Example

The directional derivative of
$f(x,y) = 3x^2y$ at the point (1, 2) in the
direction of the vector $a = 3i + 4j$.

$$f(x,y) = 3x^2y$$

$f_x(x,y) = 6xy$, $\quad f_y(x,y) = 3x^2$

so that

$f_x(1,2) = 12$, $\quad f_y(1,2) = 3$

$\mathbf{a} = 3\mathbf{i}+4\mathbf{j}$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{25}}(3\mathbf{i}+4\mathbf{j})$$

$$= \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f(1,2) = 12\left(\frac{3}{5}\right) + 3\left(\frac{4}{5}\right)$$

$$= \frac{48}{5}$$

Note:
Formula for the directional derivative can be written in the following compact form using the gradient notation

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \hat{\mathbf{u}}$$

The dot product of the gradient of f with a unit vector $\hat{\mathbf{u}}$ produces the

f_x means that function f(x,y) is differentiating partially with respect to x and
f_y means that function f(x,y) is differentiating partially with respect to y.

Example

$$f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1)$$

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$$

$$|\mathbf{u}| = \sqrt{3^2 + (-4)^2} = 5$$

$$\hat{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$f_x = 4x \qquad f_x(-1, 1) = -4$

$f_y = 2y \qquad f_y(-1, 1) = 2$

$$D_{\mathbf{u}}f(-1,1) = f_x(-1,1)u_1 + f_y(-1,1)u_2$$

$$= -\frac{12}{5} - \frac{8}{5} = -4$$

Another example, In this example we have to find directional derivative of the function $f(x,y) = 2x^2 + y^2$ at the point $P_0(-1,1)$ in the direction of $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$. **To find the directional derivative we again use the above formula**

Remarks

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector
making an angle θ with the positive
x-axis, then

$u_1 = \cos\theta$ and $u_2 = \sin\theta$

$$D_{\mathbf{u}}f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$$

can be written in the form

$$D_{\mathbf{u}}f(x_0,y_0) = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta$$

Example

The directional derivative of e^{xy} at $(-2,0)$ in the direction of the unit vector **u** that makes an angle of $\pi/3$ with the positive x-axis.

$$f(x,y) = e^{xy}$$

$f_x(x.y) = ye^{xy}$, $f_y(x, y) = x\,e^{xy}$

$fx(-2,0) = 0$, $fy(2,) = -2$

$$D_{\mathbf{u}}f(-2, 0) = f_x(-2, 0)\cos\frac{\pi}{3} + f_y(-2, 0)\sin\frac{\pi}{3}$$

$$= 0\left(\frac{1}{2}\right) + (-2)\left(\frac{\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}$$

Gradient of function

If f is a function of x and then the gradient of is defined

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

Directional Derivative

Formula for the directional derivative can be written in the following compact form using the gradient

$$D_{\mathbf{u}}\, f(x, y) = \nabla f(x, y) \cdot \hat{\mathbf{u}}$$

The dot product of the gradient f with a unit $\hat{\mathbf{u}}$ produces directional derivative of f in direction **u**.

EXAMPLE

$$\mathbf{f(x,y) = 2xy - 3y^2, \quad P_0\ (5,5)}$$

$\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$ $|\mathbf{u}| = \sqrt{4^2 + 3^2} = 5$

$$\hat{u} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$$

$f_x = 2y$, $fy = 2x - by$

$f_x(5, 5) = 10$, $fy(5,5) = -20$

$\nabla f = 10\mathbf{i} - 20\mathbf{j}$

$D_{\mathbf{u}}f(5,5) = \nabla f \cdot \hat{u}$

$$= 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right)$$

$$= -4$$

,In this example we have to find directional derivative of the function $f(x, y) = 2x\ y - 3y^2$ at the point $P_0(5,5)$ in the direction of **u = 4i + 3j. To find the directional derivative we again use the above formula**

EXAMPLE

Directional derivative of the function
$f(x, y) = xe^y + \cos(xy)$ at the point
(2,0) in the direction of $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$.

$$\hat{\mathbf{u}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$f_x(x,y) = e^y - y\sin(xy)$
$f_y(x,y) = xe^y - x\sin(xy)$
The partial derivatives of f
at (2, 0) are
$f_x(2,0) = e^0 - 0 \quad = 1$
$f_y(2,0) = 2e^0 - 2.0 = 2$

The gradient of f at (2, 0)
$\nabla f_{(2,0} = {}_x(2,0)\,\mathbf{i} + {}_y(2,0)\,\mathbf{j}$
$= \mathbf{i} + \quad \mathbf{j}$
The derivative of f at 0) in the
direction of a is
$(\ {}_{\mathbf{u}}f)_{(2,0} = \nabla f_{(2,0}\ .$

$$= \quad \mathbf{j})\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)$$

$$\frac{3}{5} - \frac{8}{5} = -$$

Properties of Directional Derivatives

$$\mathbf{D_u f = \nabla f \,.\, \hat{u} = |\nabla f| \cos\theta}$$

1. The function f increases most rapidly when $\cos\theta = 1$, or when $\hat{\mathbf{u}}$ is the direction of ∇f. That is, at each point P in its domain, **f increases most rapidly in the direction of the gradient vector ∇ f at P.** The derivative in this direction is

$$D_{\mathbf{u}} f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, ***f* decreases most rapidly in the direction of $-\nabla$f.** The derivative in this direction is

$$D_{\mathbf{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|.$$

3. **Any direction $\hat{\mathbf{u}}$ orthogonal of the gradient is a direction of zero change in f** because θ then equals $\pi/2$ and

$D_u f = |\nabla f| \cos(\pi/2) = |\nabla f| .0 = 0$

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$$

a) The direction of rapid change .

The function increases most rapidly in the direction of ∇f at (1,1). The gradient is

$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}$.

its direction is

$$\hat{\mathbf{u}} = \frac{\mathbf{i}+\mathbf{j}}{|\mathbf{i}+\mathbf{j}|}$$
$$= \frac{\mathbf{i}+\mathbf{j}}{\sqrt{(1)^2+(1)^2}}$$
$$= \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(1,1) are the directions orthogonal to ∇f.

b) **The directions of zero change**

The directions of zero change at

$$\hat{\mathbf{n}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

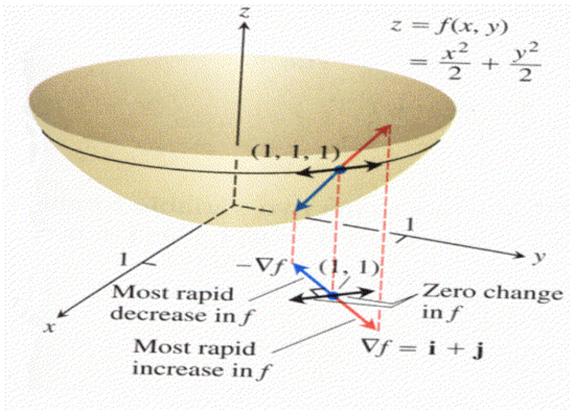
and $$-\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

The directions of rapid $-\nabla f$ at(1,1), which is

c)

The function decreases most

$\frac{1}{2}$ $\hat{\mathbf{u}} =$


z
$z = f(x, y)$
$= \frac{x^2}{2} + \frac{y^2}{2}$
(1, 1, 1)
1
y
1
x
$-\nabla f$
(1, 1)
Most rapid
decrease in f
Zero change
in f
Most rapid
increase in f
$\nabla f = \mathbf{i} + \mathbf{j}$

Lecture No -12 Tangent planes to the surfaces

Normal line to the surfaces

If **C** is a smooth parametric curve on three dimension, then tangent line to **C** at the point P_0 is the line through P_0 along the unit tangent vector to the C at the P_0.The concept of a tangent plane builds on this definition.

If $P_0(x_0,y_0,z_0)$ is a point on the Surface **S,** and if the tangent lines at P_0 to all the smooth curves that pass through P_0 and lies on the surface S all lie in a common plane, then we shall regards that plane to be the ***tangent plane*** to the surface S at P_0.

Its normal (the straight line through P_0 and perpendicular to the tangent) is called the ***surface normal*** of S at P_0.

Different forms of equation of straight line in two dimensional space

1. **Slope intercept form of the Equation of a line.**

$$y = mx + c$$

Where **m** is the slope and **c** is y intercept.

2.Point_Slope Form

Let m be the slope and $P_0(x_0, y_0)$ be the point of required line, then

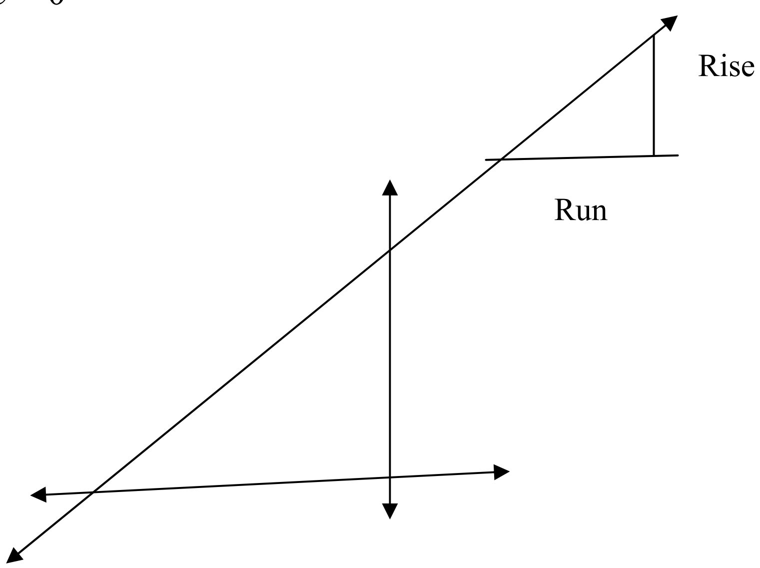
$$\mathbf{y - y_0 = m\,(x - x_0)}$$

3. General Equation of straight line

$$Ax + By + C = 0$$

$$m = \text{slope of line} = \frac{\text{Rise}}{\text{Run}} = \frac{b}{a}$$

$$y - y_0 = \frac{b}{a}\,(x - x_0)$$



Parametric equation of a line

Parametric equation of a line in two dimensional space passing through the point (x_0, y_0) and parallel to the vector a**i** + b**j** is given by

$$\mathbf{x = x_0 + at, \qquad y = y_0 + bt}$$

Eliminating t from both equation we get

$$\frac{x-x_0}{a} = \frac{y-y_0}{b}$$

$$y-y_0 = \frac{b}{a}(x-x_0)$$

Parametric vector form:

$$\mathbf{r}(t) = (x_0 + at)\,\mathbf{i} + (y_0 + bt)\,\mathbf{j},$$

Equation of line in three dimensional

Parametric equation of a line in three dimensional space passing through the point (x_0, y_0, z_0) and parallel to the vector ai + bj + ck is given by

$$\mathbf{x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct}$$

Eliminating t from these equations we get

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

EXAMPLE

Parametric equations for the straight line through the point A (2, 4, 3) and parallel to the vector $\mathbf{v} = 4\mathbf{i} + 0\mathbf{j} - 7\mathbf{k}$.

$x_0 = 2, y_0 = 4, z_0 = 3$

and a = 4, b = 0 , c = - 7.

The required parametric equations of the straight line are

$$\mathbf{x = 2 + 4t,}$$
$$\mathbf{y = 4 + 0t,}$$
$$\mathbf{z = 3 - 7t}$$

Different form of equation of curve

Curves in the plane are defined in different ways

Explicit form:

$$y = f(x)$$

Example

$$y = \sqrt{9-x^2}\ , \qquad -3 \le x \le 3.$$

Implicit form:

$$F(x, y) = 0$$

Example

]

$$x^2 + y^2 = 9, \qquad -3 \le x \le 3, \qquad 0 \le y \le 3$$

Parametric form:

$$x = f(t) \text{ and } y = g(t)$$

Example

$$x = 3\cos\theta, \qquad y = 3\sin\theta, \qquad 0 \le \theta \le \pi$$

$$x = 3\cos\theta, \quad y = 3\sin\theta$$

$$x^2 + y^2 = 9\cos^2\theta + 9\sin^2\theta$$

$$= 9(\mathrm{Cos}^2\theta + \mathrm{Sin}^2\theta)$$

$$x^2 + y^2 = 9$$

Parametric vector form:

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}, \qquad a \le t \le b.$$

$$\mathbf{r}(t) = 3\cos\theta\mathbf{i} + 3\sin\theta\mathbf{j},\ 0 \le \theta \le \pi.$$

Equation of a plane

A plane can be completely determined if we know its one point and direction of perpendicular (normal) to it.

Let a plane passing through the point $P_0\ (x_0, y_0, z_0)$ and the direction of normal to it is along the vector

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Let P (x, y, z) be any point on the plane then the line lies on it so that **n** $\perp$ $\overrightarrow{P_0P}$ ($\perp$ means perpendicular to)

$\overrightarrow{P_0P} = (x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k}$

$n \,.\, \overrightarrow{P_0P} = 0$

$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$

is the required equation of the plane

Here we use the theorem ,let a and b be two vectors, if a and b are perpendicular then a.b=0 so **n** and P_0P **a**re perpendicular vector so $n.P_0P=0$

REMARKS

Point normal form of equation of plane is

$$\mathbf{a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0}$$

We can write this equation as

$$\mathbf{ax + by + cz - ax_0 - by_0 - cz_0 = 0}$$

$$\mathbf{ax + by + cz + d = 0}$$

where $\mathbf{d = - ax_0 - by_0 - cz_0}$

Which is the equation of plane

EXAMPLE

An equation of the plane passing through the point (3, - 1, 7) and perpendicular to the vector **n** = 4**i** + 2**j** - 5**k**.

A point-normal form of the equation is

$$\mathbf{4(x - 3) + 2\,(y + 1) - 5\,(z - 7) = 0}$$

$$\mathbf{4x + 2y - 5z + 25 = 0}$$

Which is the same form of the equation of plane **ax + by + cz + d = 0**

The general equation of straight line is $ax + by + c = 0$

Let (x_1, y_1) and (x_2, y_2) be two points on this line then

$ax_1 + by_1 + c = 0$

$ax_2 + by_2 + c = 0$

Subtracting above equation

$a(x_2 - x_1) + b\,(y_2 - y_1) = 0$

$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$
is a vector in the direction of line
$\phi(x, y) = ax + by$
$\phi_x = a, \quad \phi_y = b$
$\nabla\phi = a\mathbf{i} + b\mathbf{j} = \mathbf{n}$
$\nabla\phi \,.\, \mathbf{r} = 0$
Then n and **v** are perpendicular
The general equation of plane is
$ax + by + cz + d = 0$
For any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lying on this plane we have
$ax_1 + by_1 + cz_1 + d = 0 \qquad (1)$
$a x_2 + by_2 + cz_2 + d = 0 \qquad (2)$
Subtracting equation (1) from (2) have
$a (x_2-x_1)+b (y_2-y_1) + c (z_2 - z_1) = 0$
$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).\left[(X_2-X_1)\, i+(Y_2-Y_1)\, j +(Z_2-Z_1)\, k\right]$
Here we use the definition of dot product of two vectors.

$\phi = ax + by + cz$
$\phi_x = a, \quad \phi_y = b, \quad \phi_z = c$
$\nabla\phi = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
Where $\mathbf{v} = (x_2 - x_1)\mathbf{i}+(y_2-y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$

$\nabla\phi$ is always normal to the plane.

Gradients and Tangents to Surfaces

$f (x, y) = c$
$z = f(x,y), \; z = c$

If a differentiable function f(x,y) has a constant value c along a smooth curve having parametric equation

$x = g(t), \quad y = h(t), \; r = g(t)i+h (t)j$

differentiating both sides of this equation with respect to t leads to the equation

$$\frac{d}{dt} f\,(g(t), h(t)) = \frac{d}{dt}(\,c\,)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt}+\frac{\partial f}{\partial y}\frac{dh}{dt}=0 \qquad \text{Chain Rule}$$

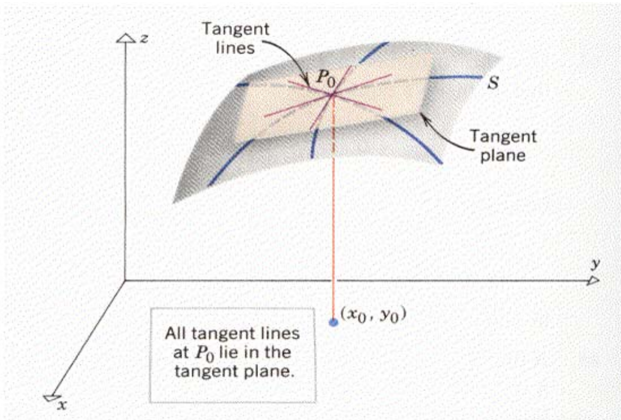
$$\left(\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}\right)\cdot\left(\frac{dg}{dt}i+\frac{dh}{dt}\boldsymbol{j}\right)=0$$

$$\nabla f.\frac{d\boldsymbol{r}}{dt}=0$$

∇f is normal to the tangent vector d **r**/dt,
so it is normal to the curve through (x_0, y_0).

Tangent Plane and Normal Line

Consider all the curves through the point
P0(x0, y0, z0) on a surface **f(x, y, z) = 0.** The plane containing all the tangents to these curves at the point P0(x0, y0, z0) is called the ***tangent plane to the surface at the point P0***.
The straight lines perpendicular to all these tangent lines at P_0 is called the ***normal line to the surface at P0*** if fx, fy, fz are all continuous at P_0 and not all of them are zero, then gradient f (i.e fx**i** + fy**j** + fz**k**) at P0 gives the direction of this normal vector to the surface at P0.



Tangent plane

Let P_0 (x_0, y_0, z_0) be any point on the Surface f(x,y,z) = 0. If f(x,y,z) is differentiable at $p_o(x_0,y_0,z_0)$ then the tangents plane at the point P_0 (x_0,y_0,z_0) has the equation

EXAMPLE

$$9x^2 + 4y^2 - z^2 = 36 \qquad P\ (2,3,6).$$

$$\boldsymbol{f(x,y,z)} = \mathbf{9x^2 + 4y^2 - z^2 - 36}$$

$$f_x = 18x, \qquad f_y = 8y, \qquad f_z = -2z$$

Equations of Tangent Plane to the surface

$$f_x(P) = 36, \quad f_y(P) = 24, \quad f_z(P) = -12$$

through P is

$$36(x-2) + 24(y-3) - 12(z-6) = 0$$
$$3x + 2y - z - 6 = 0$$

EXAMPLE

$$\mathbf{z = x\cos y - ye^x \quad (0,0,0).}$$

$\cos y - ye^x - z = 0$

$f(x,y,z) = \cos y - ye^x - z$

$f_x(0,0,0) = (\cos y - ye^x)_{(0,0)} = 1 - 0.1 = 1$

$f_y(0,0,0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$

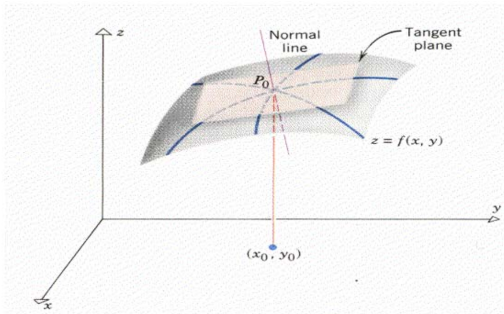
$f_z(0,0,0) = -1$

The tangent plane is

$f_x(0,0,0)(x-0) + f_y(0,0,0)(y-0) + f_z(0,0,0)(z-0) = 0$

$1(x-0) - 1(y-0) - 1(z-0) = 0,$

$x - y - z = 0.$



Lecture No -13 Orthogonal Surface

In this Lecture we will study the following topics

- **Normal line**
- **Orthogonal Surface**
- **Total differential for function of one variable**
- **Total differential for function of two variables**

Normal line

Let $P_0(x_0,y_0,z_0)$ be any point on the surface $f(x,y,z)=0$ If $f(x,y,z)$ is differentiable at $P_0(x_0,y_0,z_0)$ then the normal line at the point $P_0(x_0,y_0,z_0)$ has the equation

$$x = x_0+f_x(P_0)t, \quad y = y_0+f_y(P_0)t, \quad z = z_0+f_z(P_0)t$$

Here $\mathbf{f_x}$ means that the function f(x,y,z) is partially differentiate with respect to x And $\mathbf{f_x(P_0)}$ means that the function **f(x,y,z)** is partially differentiate with respect to x at the point $\mathbf{P_0(x_0,y_0,z_0)}$

$\mathbf{f_y}$ means that the function f(x,y,z) is partially differentiate with respect to y And $\mathbf{f_y(P_0)}$ means that the function **f(x,y,z)** is partially differentiate with respect to y at the point $\mathbf{P_0(x_0,y_0,z_0)}$

Similarly

$\mathbf{f_z}$ means that the function f(x,y,z) is partially differentiate with respect to z And $\mathbf{f_z(P_0)}$ means that the function **f(x,y,z)** is partially differentiate with respect to z at the point $\mathbf{P_0(x_0,y_0,z_0)}$

<u>EXAMPLE</u>

Find the Equation of the tangent plane and normal of the surface f(x,y,z)= x2+y2+z2-4 at the point P(1,-2,3)

$$\mathbf{f(x,y,z) = x^2+y^2+z^2 - 14}$$

$$\mathbf{P\ (1, -2, 3).}$$

$$f_x = 2x, \qquad f_y = 2y, \qquad f_z = 2z$$

$$f_x(p_0) = 2, \quad f_y(P_0) = -4, \quad f_z(P_0) = 6$$

Equation of the tangent plane to the surface at P is

$$2(x-1) - 4(y+2) + 6(z-3) = 0$$
$$x - 2y + 3z - 14 = 0$$

Equations of the normal line of the surface through P are

$$\frac{x-1}{2} = \frac{y+2}{-4} = \frac{z-3}{6}$$
$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3}$$

EXAMPLE

Find the equation of the tangent plane and normal plane

$$4x^2 - y^2 + 3z^2 = 10 \quad P(2, -3, 1)$$
$$f(x,y,z) = 4x^2 - y^2 + 3z^2 - 10$$
$$f_x = 8x, \quad f_y = -2y, \quad f_z = 6z$$
$$f_x(P) = 16, \quad f_y(P) = 6, \quad f_z(P) = 6$$

Equations of Tangent Plane to the surface through P is

$$16(x-2) + 6(y+3) + 6(z-1) = 0$$
$$8x + 3y + 3z = 10$$

Equations of the normal line to the surface through P are

$$\frac{x-2}{16} = \frac{y+3}{6} = \frac{z-1}{6}$$
$$\frac{x-2}{8} = \frac{y+3}{3} = \frac{z-1}{3}$$

Example

$$z = \frac{1}{2} x^7 y^{-2}$$
$$f(x,y,z) = \frac{1}{2} x^7 y^{-2} - z$$
$$f_x = \frac{7}{2} x^6 . y^{-2}, \quad f_y = - x^7 . y^{-3}, \quad f_z = -1$$
$$f_x(2, 4, 4) = \frac{7}{2} (2)^6 (4) = 896$$
$$f_y(2, 4, 4) = -(2)^7 (4)^{-3} = -2$$
$$f_z(2, 4, 4) = -1$$

Equation of Tangent at (2, 4, 4) is given by

$$f_x(2,4,4)(x-2) + f_y(2,4,4)(y-4) + f_z(2,4,4)(z-4) = 0$$
$$896(x-2) + (-2)(y-4) - (z-4) = 0$$
$$896x - 2y - z - 1788 = 0$$

The normal line has equations

$$x = 2 + f_x(2,4,4)t, \quad y = 4 + f_y(2,4,4)\,t, \quad z = 4 + f_z(2,4,4)t$$
$$x = 2 + 896t, \quad y = 4 - 2t, \quad z = 4 - t$$

ORTHOGONAL SURFACES

Two surfaces are said to be orthogonal at a point of their intersection if their normals at that point are orthogonal. They are Said to intersect orthogonally if they are orthogonal at every point common to them.

CONDITION FOR ORTHOGONAL SURFACES

Let (x, y, z) be any point of intersection of

$$f(x, y, z) = 0 \text{----} \quad (1)$$

$$\text{and} \quad g(x, y, z) = 0 \text{-----}(2)$$

Direction ratios of a line normal to (1) are f_x, f_y, f_z

Similarly, direction rations of a line normal to (2) are g_x, g_y, g_z

The two normal lines are orthogonal if and only if

$$f_x g_x + f_y g_y + f_z g_z = 0$$

EXAMPLE

Show that given two surfaces are orthogonal or not

$f(x,y,z) = x^2 + y^2 + z - 16$ (1)

$g(x,y,z) = x^2 + y^2 - 638$ (2)

Adding (1) and (2)

$x^2 + y^2 = \dfrac{63}{4}$, $z = \dfrac{1}{4}$ (3)

$f_x = 2x, \quad f_y = 2y \quad f_z = 1$

$g_x = 2x, \quad g_y = 2y \quad g_z = -63$

$f_x g_x + f_y g_y + f_z g_z$

$= 4(x^2 + y^2) - 63$ using (3)

$f_x g_x + f_y g_y + f_z g_z$

$= 4\left(\dfrac{63}{4}\right) - 63 = 0$

Since they satisfied the condition of orthogonality so they are orthogonal.

Differentials of a functions

For a function $y = f(x)$

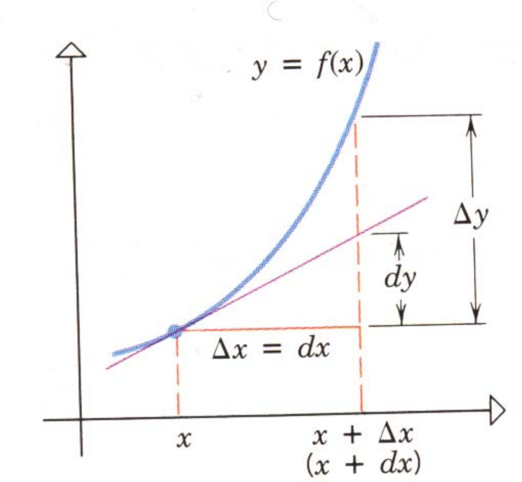
$$dy = f'(x)\,dx$$

is called the differential of functions f(x)

dx the differential of x is the same as the actual change in x

i.e. $dx = \nabla x$ where as dy the differential of y is the approximate change in the value of the functions which is different from the actual change ∇y in the value of the functions.

Distinction between the increment Δy and the differential dy



Approximation to the curve

If *f* is differentiable at x, then the tangent line to the curve y = f(x) at x_0 is a reasonably good approximation to the curve y = f(x) for value of x near x_0 Since the tangent line passes

through the point $(x_0, f(x_0))$ and has slope $f'(x_0)$, the point-slope form of its equation is

$$y - f'(x_0) = f(x_0)(x - x_0) \quad \text{or}$$

$$y = f(x_0) + f'(x_0)\,(x - x_0)$$

EXAMPLE

$$\mathbf{f(x) = \sqrt{x}}$$

$$\mathbf{x = 4 \text{ and } dx = \Delta x = 3 \quad y = \sqrt{3}}$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$= \sqrt{7} - \sqrt{4} \approx .65$$

If $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\,dx$$

$$= \frac{1}{2\sqrt{x}}\,(3) = \frac{3}{4} = .75$$

EXAMPLE

Using differentials approximation for the value of cos 61 °.

Let y = cosx and x = 60°

then dx = 61° – 60° = 1°

$$\Delta y \approx dy = -\sin x\, dx = -\sin 60° \,(1°)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

Now y = cos x

$$y+\Delta y = \cos(x+\Delta x) = \cos(x+dx)$$

$$= \cos(60°+1°) = \cos 61°$$

$$\cos 61° = y+\Delta y = \cos x + \Delta y$$

$$\approx \cos 60° - \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

$$\cos 61° \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

$$= 0.5 - 0.01511 = 0.48489$$

$$\cos 61° \approx 0.48489$$

EXAMPLE

A box with a square base has its height twice is width. If the width of the box is 8.5 inches with a possible error of ± 0.3 inches

.

Let x and h be the width and the height of the box respectively, then its volume V is given by

$$V = x^2 h$$

Since h = 2x, so (1) take the form

$$V = 2x^3$$

$$dV = 6x^2\, dx$$

Since x = 8.5, dx = ±0.3, so putting these values in (2), we have

$$dV = 6\,(8.5)^2\,(\pm 0.3) = \pm 130.05$$

This shows that the possible error in the volume of the box is ±130.05.

TOTAL DIFFERENTIAL

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting differential in f is

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

This change in the linearization of

f is called the total differential of f.

EXACT CHANGE

Area = xy

x = 10, y = 8 Area = 80
x = 10.03 y = 8.02 Area = 80.4406
Exact Change in area = 80.4406 − 80
= 0.4406

EXAMPLE

A rectangular plate expands in such a way that its length changes from 10 to 10.03 and its breadth changes from 8 to 8.02.

Let x and y the length and breadth of the rectangle respectively, then its area is

$$A = xy$$

$$dA = A_x\,dx + A_y dy = ydx + xdy$$

By the given conditions

$$x = 10,\ dx = 0.03,\ y = 8,\ dy = 0.02.$$

$$dA = 8(0.03) + 10(0.02) = 0.44$$

Which is exact Change

EXAMPLE

The volume of a rectangular parallelepiped is given by the formula V = xyz. If this solid is compressed from above so that z is decreased by 2% while x and y each is increased by 0.75% approximately

$$V = xyz$$

$$dV = V_x d\,x + V_y dy + V_z dx$$

$$dV = yzdx + xzdy + zydz \quad (1)$$

$$dx = \frac{0.75}{100}x,\ dy = \frac{0.75}{100}y,\ dz = -\frac{2}{100}z$$

Putting these values in (1), we have

$$dV = \frac{0.75}{100}xyz + \frac{0.75}{100}xyz - \frac{2}{100}xyz$$

$$= -\frac{0.5}{100}xyz = -\frac{0.5}{100}V$$

This shows that there is 0.5 % decrease in the volume.

EXAMPLE

A formula for the area Δ of a triangle is $\Delta = \frac{1}{2}$ ab sin C. Approximately what error is made in computing Δ if a is taken to be 9.1 instead of 9, b is taken to be 4.08 instead of 4 and C is taken to be $30°3'$ instead of $30°$.

By the given conditions

$$a = 9,\ b = 4,\ C = 30°,$$
$$da = 9.1 - 9 = 0.1,$$
$$db = 4.08 - 4 = 0.08$$
$$dC = 30°3' - 3' = \left(\frac{3}{60}\right)^{\circ}$$
$$= \frac{3}{60} \times \frac{\pi}{180} \text{ radians}$$

Putting these values in (1), we have

$$\Delta = \frac{1}{2} \text{ ab sin C}$$
$$d\Delta = \frac{\partial}{\partial a}\left(\frac{1}{2}\text{ab sin C}\right)da + \frac{\partial}{\partial b}\left(\frac{1}{2}\text{ab sin C}\right)db$$
$$+ \frac{\partial}{\partial C}\left(\frac{1}{2}\text{ab sin C}\right)dC$$
$$d\Delta = \frac{1}{2}\text{ b sin Cda} + \frac{1}{2}\text{ a sin C db}$$
$$+ \frac{1}{2}\text{ ab cos CdC}$$

$$d\Delta = \frac{1}{2}4 \sin 30°(0.1) + \frac{1}{2}9 \sin 30°(0.08)$$
$$+ \frac{1}{2}36 \cos 30°\left(\frac{\pi}{3600}\right)$$
$$d\Delta = 2\left(\frac{1}{2}\right)(0.1) + \frac{1}{2}\left(\frac{1}{2}\right)(0.08)$$
$$+18\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3.14}{3600}\right) = 0.293$$
$$\% \text{ change in area} = \frac{0.293}{\Delta} \times 100$$
$$= \frac{0.293}{9} \times 10 = 3.25\%$$

Lecture No -14 Extrema of Functions of Two Variables

In this lecture we shall fined the techniques for finding the highest and lowest points on the graph of a function or , equivalently, the largest and smallest values of the function.

The graph of many functions form hills and valleys. The tops of the hills are relative maxima and the bottom of the valleys are called relative minima. Just as the top of a hill on the earth's terrain need not be the highest point on the earth , so a relative maximum need not be the highest point on the entire graph .

Absolute maximum

A function f of two variables on a subset of $\mathbf{R}^2$ is said to have an D **absolute (global) maximum** value on D if there is some point (x_0, y_0) of D such that value of f on D

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y) \in D$$

In such a case $f(x_0, y_0)$ is the **absolute maximum**

Relative extremum and absolute extremum

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a relative extremum at (x_0, y_0), and if f has an absolute maximum or absolute minimum at $(x_0 y_0)$, then we say that f has an absolute extremum at (x_0, y_0).

Absolute minimum

A function f of two variables on a subset D of $\mathbf{R}^2$ is said to have an **absolute (global) minimum** value on D if there is some point (x_0, y_0) of D such that

$$f(x_0, y_0) \leq f(x, y) \text{ for all } (x, y) \in D.$$

In such a case $f(x_0, y_0)$ is the **absolute minimum** value of f on D.

Relative (local) maximum

The function f is said to have a relative (local) maximum at some point (x0,y0) of its domain D if there exists an open disc *K centered* at (x0,y0) and of radius r

$$K = \{(x, y) \in \mathbf{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

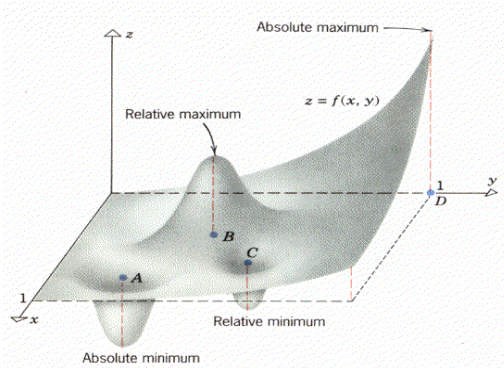
With $K \in D$ such that

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y)$$

Relative (local) minimum

The function f is said to have a **relative(local) minimum** at some point (x_0, y_0) of **D** if there exists an open disc K centred at (x_0, y_0) and of radius r with $K \subset D$ such that

$$f(x_0, y_0) \le f(x, y) \text{ for all } (x, y) \in K.$$



Extreme Value Theorem

If f (x, y) is continuous on a closed and bounded set R, then f has both an absolute maximum and on absolute minimum on R .

Remarks

If any of the conditions the Extreme Value Theorem fail to hold, then there is no guarantee that an absolute maximum or absolute minimum exists on the region R. Thus, a discontinuous function on a closed and bounded set need not have any absolute extrema, and a continuous function on a set that is not closed and bounded also need not have any absolute extrema.

Extreme values or extrema of *f*

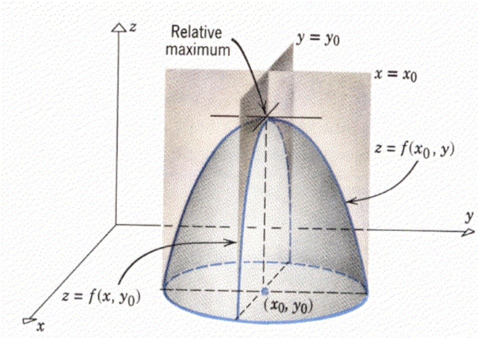
The maximum and minimum values of *f are* referred to as extreme values of extrema of f .Let a function f of two variables be defined on an open disc

$$K = \{(x, y): (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

Suppose $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist on K

If f has relative extrema at (x_0,y_0),then

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0).$$



Saddle Point

A differentiable function f(x, y) has a saddle has a saddle point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where f (x, y) > f (a, b) and domain points (x, y) where f (x, y) < f (a, b). The corresponding point (a, b, f (a, b)) on the surface z = f (x, y) is called a saddle point of the surface

Remarks

Thus, the only points where a function f(x,y) can assume extreme values are critical points and boundary points. As with differentiable functions if a single variable, not every critical point gives rise to o a local extremum. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variable might have a saddle point.

EXAMPLE

Fine the critical points of the given function

$$\boldsymbol{f(x, y) = x^3 + y^3 - 3axy, a > 0.}$$

f_x, f_y exist at all points of the domain of f.

$f_x = 3x_2 - 3ay, \quad f_y = 3y^2 - 3ax$

For critical points $f_x = f_y = 0$.

Therefore, $x^2 - ay = 0 \quad (1)$

and $\quad ax - y^2 = 0 \quad (2)$

Substituting the value of x from (2) into (1), we have

$$\frac{y^4}{a^2} - ay = 0$$

$$y(y^3 - a^3) = 0$$

$y = 0, \qquad y = a$

and so

$x = 0, \qquad x = a$.

The critical points are (0, 0) and (a, a).

Overview of lecture # 14

Book

CALCULUS by HOWARD ANTON

Lecture No - 15 Example

EXAMPLE

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

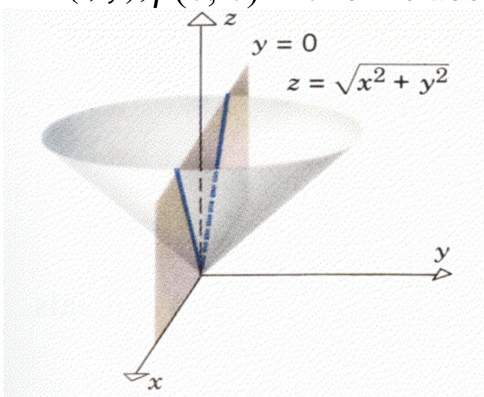
$$f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

The partial derivatives exist at all points of the domain of f except at the origin which is in the domain of f. Thus (0, 0) is a critical point of f

Now $f_x(x, y) = 0$ only if $x = 0$ and
$f_y(x, y) = 0$ only if $y = 0$

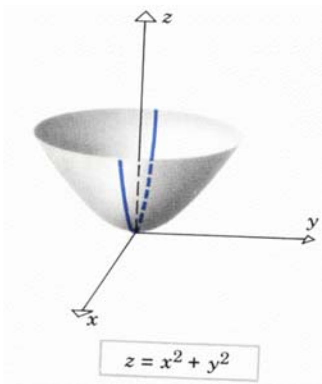
The only critical point is (0,0) and f(0,0)=0

Since $f(x, y) \geq 0$ for all (x, y), $f(0, 0) = 0$ is the absolute minimum value of f.



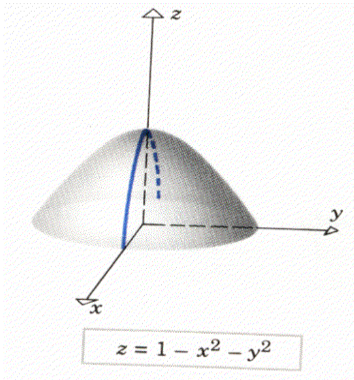
Example

$$z = f(x, y) = x^2 + y^2 \text{ (Paraboloid)}$$
$$f_x(x, y) = 2x, \quad f_y(x, y) = 2y$$
when $f_x(x, y) = 0$, $f_y(x, y) = 0$
we have (0, 0) as critical point.



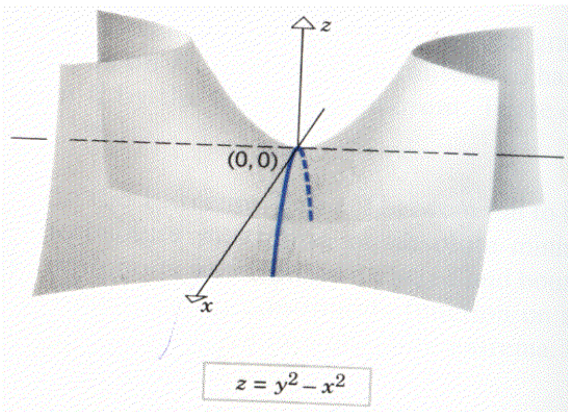
EXAMPLE

$$z = g(x, y) = 1 - x^2 - y^2 \text{ (Paraboloid)}$$
$$g_x(x, y) = -2x, \quad g_y(x, y) = -2y$$
$$\text{when } g_x(x, y) = 0, \quad g_y(x, y) = 0$$
we have (0, 0) as critical point.



EXAMPLE

$$z = h(x,y) = y^2 - x^2 \text{ (Hyperbolic paraboloid)}$$
$$h_x(x, y) = -2x, \quad h_y(x, y) = 2y$$
$$\text{when } h_x(x, y) = 0, \quad h_y(x, y) = 0$$
we have (0, 0) as critical point.



EXAMPLE

$$\mathbf{f(x, y) = \sqrt{x^2 + y^2}}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} \quad f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

The point (0,0) is critical point of f because the partial derivatives do not both exist. It is evident geometrically that $f_x(0,.0)$ does not exist because the trace of the cone in the plane y=0 has a corner at the origin.

The fact that $f_x(0,0)$ does not exist can also be seen algebraically by noting that $f_x(0,0)$ can be interpreted as the derivative with respect to x of the function

$$f(x, 0) = \sqrt{x^2 + 0} = |x| \quad \text{at } x = 0.$$

But |x| is not differentiable at x = 0, so $f_x(0,0)$ does not exist. Similarly, $f_y(0,0)$ does not exist. The function f has a relative minimum at the critical point (0,0).

The Second Partial Derivative Test

Let *f* be a function of two variables with continuous second order partial derivatives in some circle centered at a critical point (x_0, y_0), and let

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0)$$

(a) If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then f has a **relative minimum** at (x_0,y_0).

(b) If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then f has a **relative maximum** at (x_0,y_0).

(c) If $D < 0$, then f has a **saddle point** at (x_0,y_0).

(d) If $D = 0$, then no conclusion can be drawn.

REMARKS

If a function *f* of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.

EXAMPLE

$$f(x,y) = 2x^2 - 4x + xy^2 - 1$$

$f_x(x, y) = 4x - 4 + y^2$, $\quad f_{xx}(x, y) = 4$

$f_y(x, y) = 2xy$, $\quad f_{yy}(x, y) = 2x$

$f_{xy}(x, y) = f_{yx}(x, y) = 2y$

For critical points, we set the first partial derivatives equal to zero. Then

$4x - 4 + y^2 = 0 \quad (1)$

and $2xy = 0 \quad (2)$

we have $x = 0$ or $y = 0$

$x = 0$, then from (1), $y = \pm 2$.

$y = 0$, then from (1), $x = 1$.

Thus the critical points are **(1,0), (0, 2), (0, − 2).**

We check the nature of each point.

$f_{xx}(1,0) = 4,$

$f_{yy}(1, 0) = 2,$

$f_{xy}(1, 0) = 0$

$D = f_{xx}(1, 0).f_{yy}(1, 0) - [f_{xy}(1, 0)]^2$

$= 8 > 0$

and $f_{xx}(1, 0)$ is positive. Thus f has a relative minimum at (1, 0).

$f_{xx}(0,-2) = 4,$

$f_{yy}(0,-2) = 0,$

$f_{xy}(0,-2) = -4$

$D = f_{xx}(0, -2).f_{yy}(0, -2) - [f_{xy}(0, -2)]^2$

$= -16 < 0.$ $f_{xx}(0, 2) = 4,$

$f_{yy}(0, 2) = 0,$

$f_{xy}(0, 2) = 4$

$D = f_{xx}(0, 2).f_{yy}(0, 2) - [f_{xy}(0, 2)]^2$

$= -16 < 0.$

Therefore, f has a saddle point at (0,2).

Therefore, f has a saddle point at (0, − 2).

EXAMPLE

$$f(x,y) = e^{-(x^2+y^2+2x)}$$

$$f_x(x, y) = -2(x+1)e^{-(x^2+y^2+2x)},$$

$$f_y(x, y) = -2ye^{-(x^2+y^2+2x)}$$

For critical points

$f_x(x,y) = 0, \quad x + 1 = 0, \quad x = -1$ and

$f_y(x, y) = 0, \quad y = 0$

Hence critical point is (−1,0).

$$f_{xx}(x,y) = [(-2x-2)^2 - 2]e^{-(x^2+y^2+2x)}$$

$$f_{xx}(-1, 0) = -2,$$

$$f_{yy}(x,y) = [4y^2 - 2]e^{-(x^2+y^2+2x)}$$

$$f_{yy}(-1, 0) = -$$

$$f_{xy}(x,y) = -2y(-2x-2)e^{-(x^2+y^2+2x)}$$

$$f_{xy}(-1, 0) = 0$$

$$\mathbf{D = f_{xx}(-1,0)\, f_{yy}(-1, 0) - f^2_{xy}(-1, 0)}$$

$$= (-2e)(-2e) > 0$$

This shows that f is maximum at (−1, 0).

EXAMPLE

$$f(x,y) = 2x^4 + y^2 - x^2 - 2y$$

$f_x(x, y) = 8x^3 - 2x$, $f_y(x, y) = 2y - 2$

$f_{xx}(x, y) = 24x^2 - 2$, $f_{yy}(x,y) = 2$,

$f_{xy}(x, y) = 0$

For critical points

$f_x(x, y) = 0$,

$2x(4x^2 - 1) = 0$, $x = 0, 1/2, -1/2$

$f_y(x, y) = 0$,

$2y - 2 = 0$, $y = 1$

Solving above equation we have the critical points $(0,1), \left(-\frac{1}{2}, 1\right) \left(\frac{1}{2}, 1\right)$.

$f_{xx}(0,1) = -2$, $f_{yy}(0, 1) = 2$,

$f_{xy}(0, 1) = 0$

$\mathbf{D = f_x(0, 1)\, f_{yy}(0, 1) - f^2_{xy}(0, 1)}$

$= (-2)(2) - 0 = -4 < 0$

This shows that (0, 1) is a saddle point.

$f_{xx}\left(\frac{1}{2}, 1\right) = 4$, $f_{yy} = \left(\frac{1}{2}, 1\right) = 2$

$f_{xy}\left(\frac{1}{2}, 1\right) = 0$

$\mathbf{D = f_{xx}\left(\frac{1}{2},1\right) f_{yy}\left(\frac{1}{2},1\right) - f^2_{xy}\left(-\frac{1}{2},1\right)}$

$= (4)(2) - 0 = 8 > 0$

$f_{xx}\left(\frac{1}{2},1\right) = 4 > 0$, so f is minimum at $\left(\frac{1}{2},1\right)$.

Example

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4.$$

$f_x(x, y) = 4y - 4x^3$, $\quad f_y(x, y) = 4x - 4y^3$

For critical points

$f_x(x, y) = 0$

$4y - 4x^3 = 0 \qquad (1)$

$y = x^3$

$f_y(x, y) = 0$

$4x - 4y^3 = 0 \qquad (2)$

$x = y^3$

Solving (1) and (2), we have the critical points (0,0), (1, 1),(−1, −1).

Now $f_{xx}(x, y) = -12x^2$, $\quad f_{xx}(0, 0) = 0$

$f_{yy}(x, y) = -12y^2$, $f_{yy}(0,0) = 0$

$f_{xy}(x, y) = 4$, $\qquad f_{xy}(0, 0) = 4$

$\mathbf{D = f_{xx}(0,0)\, f_y(0,0) - f^2_{xy}(0,0)}$

$= (0)(0) - (4)^2 = -16 < 0$

This shows that (0,0) is the saddle point.

$f_{xx}(x, y) = -12x^2$, $\quad f_{xx}(1,1) = -12 < 0$

$f_{yy}(x,y) = -12y^2$, $\quad f_y(1,1) = -12$

$f_{xy}(x, y) = 4$, $\qquad f_{xy}(1,1) = 4$

$\mathbf{D = f_{xx}(1,1)\, f_{yy}(1,1) - f^2_{xy}(1, 1)}$

$= (-12)(-12) - (4)^2 = 128 > 0$

This shows that f has relative maximum at (1,1).

$f_{xx}(x,y) = -12x^2$, $\quad f_{xx}(-1, -1) = -12 < 0$

$f_y(x, y) = -2y^2$, $\quad f_y(-1, -1) = -12$

$f_{xy}(x, y) = 4$, $\qquad f_{xy}(-1, -1) = 4$

$\mathbf{D = f_{xx}(-1,-1)\, f_{yy}(-1,-1) - f^2_{xy}(-1,-1)}$

$= (-12)(-12) - (4)^2 = 128 > 0$

This shows that f has relative maximum (−1, − 1).

Over view of lecture # 15 Book

Calculus by HOWARD ANTON

Topic #	**Article #**	**Page #**
Example	3	836
Graph of f(x,y)	16.9.4	836
The Second Partial Derivative Test	16.9.5	836
Example	5	837

Lecture No -16 Extreme Valued Theorem

EXTREME VALUED THEOREM

If the function f is continuous on the closed interval [a, b], then f has an absolute maximum value and an absolute minimum value on [a, b]

Remarks

An absolute extremum of a function on a closed interval must be either a relative extremum or a function value at an end point of the interval. Since a necessary condition for a function to have a relative extremum at a point C is that C be a critical point, we may determine the absolute maximum value and the absolute minimum value of a continuous function f on a closed interval [a, b] by the following procedure.

1. Find the critical points of f on [a, b] and the function values at these critical.
2. Find the values of f (a) and f (b).
3. The largest and the smallest of the above calculated values are the absolute maximum value and the absolute minimum value respectively

Example

Find the absolute extrema of

$$f(x)= x^3+ x^2-x+1 \quad \text{on} \quad [-2,1/2]$$

Since f is continuous on **[-2,1/2],** the extreme value theorem is applicable. For this

$$f'(x) =3 x^2+2x-1$$

This shows that f(x) exists for all real numbers, and so the only critical numbers of **f** will be the values of x for which $f'(x)=0$. .

Setting $f'(x) = 0$, we have

$(3x - 1) (x + 1) = 0$

from which we obtain

$x = -1$ and $x = \frac{1}{3}$

The critical points of f are -1 and $\frac{1}{3}$, and each of these points is in the given closed interval $(-2, \frac{1}{2})$ We find the function values at the critical points and at the end points of the interval, which are given below.

$f(-2) = -1, \; f(-1) = 2,$

$f\left(\frac{1}{3}\right) = \frac{22}{27}, \; f\left(\frac{1}{2}\right) = \frac{7}{8}$

The absolute maximum value of f on $(-2, \frac{1}{2})$ is therefore 2, which occurs at -1, and the absolute min. value of f on $(-2, \frac{1}{2})$ is -1, which occurs at the left end point -2.

Find the absolute extrema of

$$f(x) = (x \quad 2)^{2/3} \qquad \text{on } [1, 5].$$

Since f is continuous on [1. 5], the extreme-value ttheorem is applicable.

Differentiating f with respect to x, we get

$$f'(x) = \frac{2}{3\,(x-2)^{1/3}}$$

There is no value of x for which $f'(x) = 0$. However, since $f'(x)$ does not exist at 2,

we conclude that 2 is a critical point of f,

so that the absolute extrema occur either at 2 or at one of the end points of the interval. The function values at these points are given below.

$f(1) = 1$, $f(2) = 0$, $f(5)= \sqrt[3]{9}$

From these values we conclude that the absolute minimum value of f on [1,5] is 0, occurring at 2, and the absolute maximum value of f on [1, 5] is $\sqrt[3]{9}$,occurring at 5.

Find the absolute extrema of

$$h(x) = x^{2/3} \text{ on } [-2, 3].$$

$$h'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$$

$h'(x)$ has no zeros but is undefined at x = 0.

The values of h at this one critical point

and at the endpoints x =− 2 and x = 3 are

$h(0) = 0$

$h(-2) = (-2)^{2/3} = 4^{1/3}$

$h(3) = (3)^{2/3} = 9^{1/3}$.

The absolute maximum value is $9^{1/3}$ assumed at x = 3; the absolute minimum is 0, assumed at x = 0.

<u>How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Region R.</u>

Step 1.

Find the critical points of f that lie in the interior of R.

Setp 2.

Find all boundary points at which the absolute extrema can occur,

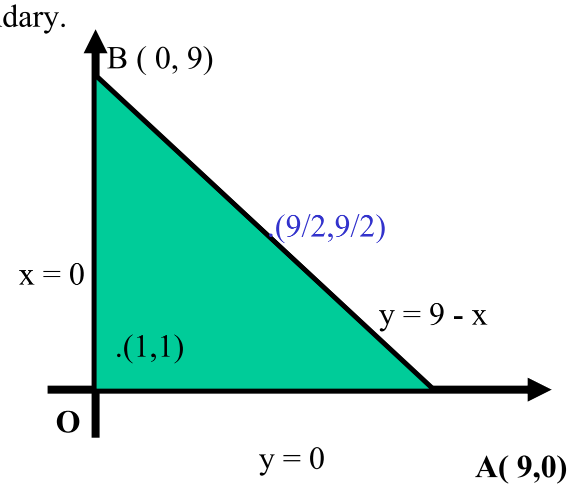
Step 3.

Evaluate f(x,y) at the points obtained in the previous steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Find the absolute maximum and minimum value of

$f(x,y) = 2 + 2x + 2y - x^2 - y^2$

On the triangular plate in the first quadrant bounded by the lines x=0,y=0,y=9-x
Since f is a differentiable, the only places where f can assume these values are points inside the triangle having vertices at O(0,0), A(9,0)and B(0,9) where $f_x = f_y = 0$ and points of boundary.



For interior points:

We have $f_x = 2-2x = 0$ and $f_y = 2-2y = 0$
yielding the single point (1,1)

For boundary points we take take the triangle one side at time :

1. On the segment OA, y=0
   $U(x) = f(x, 0) = 2+2x-x^2$

may be regarded as function of x defined on the closed interval $0 \le x \le 9$ Its extreme values may occur at the endpoints x=0 and x=9 which corresponds to points (0, 0) and (9, 0) and U(x) has critical point where

$U'(x) = 2-2x = 0$ Then x=1

On the segment OB, x=0 and
$V(y) = f(0,y) = 2+2y-y^2$

Using symmetry of function f, possible points are (0,0),(0,9) and (0,1)

3. The interior points of AB.
With y = 9 - x, we have
$f(x, y) = 2+2x+2(9-x)-x^2-(9-x)^2$
$W(x) = f(x, 9-x) = -61 + 18x - 2x^2$
Setting $w'(x) = 18 - 4x = 0$, $x = 9/2$.
At this value of x,y = 9 – 9/2
Therefore we have $\left(\frac{9}{2}, \frac{9}{2}\right)$ as a critical point.

(x, y)	(0,0)	(9,0)	(1, 0)	$\left(\frac{9}{2}, \frac{9}{2}\right)$
f(x,y)	2	–61	3	$-\frac{41}{2}$

(x, y)	(0, 9)	(0,1)	(1,1)
f(x,y)	– 61	3	4

The absolute maximum is 4 which f assumes at the point (1,1) The absolute minimum is -61 which f assumes at the points (0, 9) and (9,0)

EXAMPLE
Find the absolute maximum and the absolute minimum values of

$$f(x,y)=3xy-6x-3y+7$$

on the closed triangular region **r** with the vertices (0,0), (3,0) and (0,5) .

$$f(x, y) = 3xy - 6x - 3y + 7$$
$$f_x(x, y) = 3y - 6, \quad f_y(x, y) = 3x - 3$$
For critical points
$$f_x (x, y) = 0$$
$$3y - 6 = 0$$
$$y = 2$$
$$f_y(x, y) = 0$$
$$3x - 3 = 0$$
$$x = 1$$

Thus, (1, 2) is the only critical point in the interior of R. Next, we want to determine the location of the points on the boundary of R at which the absolute extrema might occur. The boundary extrema might occur. The boundary each of which we shall treat separately.

i)The line segment between (0, 0) and (3, 0):

On this line segment we have y=0 so (1) simplifies to a function of the single variable x,

$$u(x)=f(x, 0) = -6x + 7, 0 \leq x \leq 3$$

This function has no critical points because $u^{/}(6)=-6$ is non zero for all x . Thus, the extreme values of u(x) occur at the endpoints x = 0 and x=3 , which corresponds to the points (0, 0) and (3,0) on R

ii) The line segment between the (0,0) and (0,5)

On this line segment we have x=0 ,so single variable y,

$$v(y) = f(0, y) = -3y + 7, \ 0 \leq y \leq 5$$

This function has no critical points because $v^{/}(y)=-3$ is non zero for all y.Thus ,the extreme values of v(y) occur at the endpoints y = 0 and y=5 which correspond to the point (0,0) and (0,5) or R

iii) The line segment between (3,0) and (0,5)

In the XY- plan , an equation for the line segment

$$y = -\frac{5}{3}x + 5,\ 0 \le x \le 3$$

so (1) simplifies to a function of the single variable x,

$$w(x)=f\left(x, -\frac{5}{3}x + 5\right)$$
$$= -5x^2+14x-8,\quad 0 \le x \le 3$$
$$w'(x) = -10x + 14$$
$$w'(x) = 0$$
$$10x + 14 = 0$$
$$x = \frac{7}{5}$$

This shows that x =7/5 is the only critical point of w. Thus, the extreme values of w occur either at the critical point x=7/5 or at the endpoints x=0 and x=3.The endpoints correspond to the points (0, 5) and (3, 0) of R, and from (6) the critical point corresponds to [7/5,8/3]

(x, y)	(0, 0)	(3, 0)	(0, 5)	$\left(\frac{7}{5}, \frac{8}{3}\right)$	(1, 2)
f(x,y)	7	− 11	− 8	$-\frac{9}{5}$	1

Finally, table list the values of f(x,y) at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is f(0,0)=7 and the absolute minimum values is f(3,0)=-11.

OVER VIEW:

Maxima and Minima of functions of two variables. Page # 833

Exercise: 16.9 Q #26,27,28,29.

Lecture No -17 Examples

EXAMPLE

Find the absolute maximum and minimum values of f(x,y)=xy-x-3y on the closed triangular region R with vertices (0, 0), (0, 4), and (5, 0).

$$\boldsymbol{f}(\mathbf{x,y}) = \mathbf{xy - x - 3y} \qquad \mathbf{(1)}$$

$f_x(x, y) = y - 1, \quad f_y(x, y) = x - 3$

For critical points

$f_x(x, y) = 0, \; y - 1 = 0$

$$y = 1 \qquad (2)$$

$f_y(x, y) = 0, \; 3x - 3 = 0$

$$x = 3 \qquad (3)$$

Thus, (3, 1) is the only critical point in the interior of R. Next, we want to determine the location of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we shall treat separately.

(i) The line segment between (0, 0) and (5,0)

On this line segment we have y = 0, so (1) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -x, \; 0 \le x \le 5 \qquad (4)$$

The function has no critical points because the $u'(x)=-1$ is non zero for all x.Thus,the exteme values of u(x) occure at the endpoints x=0 and x=5 , which corresponds to the points (0,0) and (5,0) of R.

ii) The line segment between (0,0) and (0,4)

On this line segment we have x = 0, so (1) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y, \; 0 \le y \le 4. \qquad (5)$$

This function has no critical points because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of v(y) occur at the endpoints y =0 and y=4 ,which correspond to the point (0,0) and (0,4) or R.

iii) The line segment between (5,0) and (0,4)

In the xy-plan, an equation is

$$\mathbf{y} = -\frac{4}{5}x + 4, \; 0 \le x \le 5 \qquad (6)$$

so (1) simplifies to a function of the single variable x,

$$w(x) = f\left(x, -\frac{4}{5}x + 4\right)$$

$$= -\frac{4}{5}x^2 + \frac{3}{5}x + 12, \; 0 \le x \le 5$$

$$w'(x) = -\frac{8}{5}x + \frac{3}{5}, \; w'(x) = 0$$

$$x = \frac{3}{8}$$

This shows that $x = \frac{3}{8}$ is the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{3}{8}$ or at the endpoints x = 0 and x = 5. The endpoints correspond to the points (0, 4) and (5, 0) of R, and from (6) the critical point corresponds to $\left[\frac{3}{8}, \frac{37}{10}\right]$

(x, y)	(0,0)	(5,0)	(0, 4)	$\left(\frac{3}{8}, \frac{37}{10}\right)$	(3, 1)
f(x,y)	0	− 5	− 4	$-\frac{807}{80}$	− 3

Finally, from the table below, we conclude that the absolute maximum value of f is f (0,0) = 0 and the absolute minimum value is

$f\left(\frac{3}{8}, \frac{37}{10}\right)$=-807/80

Example

Find three positive numbers whose sum is 48 and such that their product is as large as possible

Let x,y and z be the required numbers, then we have to maximize the product

$$f(x,y)=xy(48-x-y)$$

Since

$$f_x=48y-2xy-y^2, \qquad f_y=48x-2xy-x^2$$

solving

$$f_x=0, \qquad f_y=0$$

we get x=16, y=16, z=16

Since x+y+z=48

$f_{xx}(x,y) = -2y$, $f_{xx}(16, 16) = -32 < 0$

$f_{xy}(x, y) = 48-2x-2y$, $f_{xy}(16, 16) = -16$

$f_{yy}(x, y) = -2x$, $f_{yy}(16, 16) = -32$

$D=f_{xx}(16,16)f_{yy}(16,16)-f^2_{xy}(16,16)$

$= (-32)(-32) - (16)^2 = 768 > 0$

For x = 16, y = 16 we have z = 16 since x + y + z = 48

Thus, the required numbers are 16, 16, 16.

Example

Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible

Let x, y, z be the required numbers, then we have to

$$f(x,y) = x^2 + y^2 + z^2$$
$$= x^2 + y^2 + (27 - x - y)^2$$

Since x+y+z = 27

$f_x = 4x+2y-54$, $f_y = 2x+4y-54$,

$f_{xx} = 4$, $f_{yy} = 4$, $f_{xy} = 2$

Solving $f_x = 0$, $f_y = 0$
We get x = 9, y = 9, z = 9
Since x + y + z = 27
$D = f_{xx}(9, 9)\, f_{yy}(9, 9) - [f_{xy}(9, 9)]^2$
$= (4)(4) - 2^2 = 12 > 0$
This shows that f is minimum
x = 9, y = 9, z = 9, so the required
numbers are 9, 9, 9.

Example

Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius 4.
Solution:
The volume of the parallelepiped with dimensions x, y, z is

$$V = xyz$$

Since the box is inscribed in the sphere of radius 4, so equation of sphere is $x^2 + y^2 + z^2 = 4^2$ from this equation we can write $z = \sqrt{16 - x^2 - y^2}$ and putting this value of "z" in above equation we get $V = xy\sqrt{16 - x^2 - y^2}$ .Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function "V". For extreme values we will find out the critical points and for critical points we will solve the equations $V_x = 0$ and $V_y = 0$ .Now we have

$$V_x = y\sqrt{16 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{16 - x^2 - y^2}}$$

$$\Rightarrow V_x = y\left\{\frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} \quad Now\ V_x = 0 \Rightarrow y\left\{\frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = 16 ....(a)$$

Similarly we have

$$V_y = x\sqrt{16 - x^2 - y^2} + \frac{xy(-2y)}{2\sqrt{16 - x^2 - y^2}}$$

$$\Rightarrow V_y = x\left\{\frac{-x^2 - 2y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} \quad Now\ V_y = 0 \Rightarrow x\left\{\frac{-x^2 - 2y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -x^2 - 2y^2 + 16 = 0 \Rightarrow x^2 + 2y^2 = 16 ....(b)$$

Solving equations (a) and (b) we get the $x = \frac{4}{\sqrt{3}}$ and $y = \frac{4}{\sqrt{3}}$

Now $V_{xx} = \dfrac{xy(2x^2 + 3y^2 - 48)}{(16 - x^2 - y^2)^{\frac{3}{2}}}$ (We obtain this by using quotient rule of differentiation)

$$V_{xx}\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{16}{\sqrt{3}} \prec 0$$

Also we have to calculate $V_{yy} = \frac{xy(3x^2+2y^2-48)}{(16-x^2-y^2)^{\frac{3}{2}}}$ and $V_{yy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}) = -\frac{16}{\sqrt{3}} \prec 0$ Also note that $V_{xy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}) = -\frac{8}{\sqrt{3}}$ Now as we have the formula for the second order partial derivative is $f_{xx}.f_{yy} - (f_{xy})^2$ and putting the values which we calculated above we note that $f_{xx}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}).f_{yy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}) - (f_{xy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}))^2 = +\frac{320}{3} \succ 0$ Which shows that the function V has maximum value when $x = \frac{4}{\sqrt{3}}$ *and* $y = \frac{4}{\sqrt{3}}$. So the dimension of the rectangular box are $x = \frac{4}{\sqrt{3}}, y = \frac{4}{\sqrt{3}}$ *and* $z = \frac{4}{\sqrt{3}}$.

Example

A closed rectangular box with volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing Rs. 10 per square foot and the sides from material costing Rs.5 per square foot. Find the dimensions of the box so that the cost of materials is minimized

Let x, y, z, and C be the length, width, height, and cost of the box respectively. Then it is clear form that

C=10(xy+xy)+5(xz+xz)+5(yz+yz)---------------(1)

C=20xy+10(x+y)z

The volume of the box is given by

xyz=16----------------------------------(2)

Putting the value of z from (2) in (1), we have

$C = 20xy + 10(x+y)\frac{16}{xy}$

$C = 20xy + \frac{160}{y} + \frac{160}{x}$

$C_x = 20y - \frac{160}{x^2}$, $C_y = 20x - \frac{160}{y^2}$

For critical points

$C_x = 0$

$20y - \frac{160}{x^2} = 0$ and $C_y = 0$

$20x - \frac{160}{y^2} = 0$

Solving these equations, we have x = 2, y = 2. Thus the critical point is (2, 2).

$C_{xx}(x, y) = \dfrac{320}{x^3}$

$C_{xx}(2, 2) = \dfrac{320}{8} = 40 > 0$

$C_{yy}(x, y) = \dfrac{320}{y^3}$

$C_{yy}(2, 2) = \dfrac{320}{8} = 40$

$C_{xy}(x, y) = 20$

$C_{xy}(2, 2) = 20$

$C_{xx}(2,2)\, C_{yy}(2,2) - C^2_{xy}(2,2) = (40)(40)-(20)^2 = 1200 > 0$

This shows that S has relative minimum at x = 2 and y = 2. Putting these values in (2), we have z = 4, so when its dimensions are 2 ×2 ×4.

Example

Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius a.

Solution:

The volume of the parallelepiped with dimensions x, y, z is

$$V = xyz$$

Since the box is inscribed in the sphere of radius 4, so equation of sphere is

$x^2 + y^2 + z^2 = 4^2$ from this equation we can write $z = \sqrt{a^2 - x^2 - y^2}$ and putting this value of "z" in above equation we get $V = xy\sqrt{a^2 - x^2 - y^2}$ .Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function "V". For extreme values we will find out the critical points and for critical points we will solve the equations $V_x = 0$ and $V_y = 0$ .Now we have

$$V_x = y\sqrt{a^2 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{a^2 - x^2 - y^2}}$$

$$\Rightarrow V_x = y\left\{\frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} \quad Now\ V_x = 0 \Rightarrow y\left\{\frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Similarly we have

$$V_y = x\sqrt{a^2 - x^2 - y^2} + \frac{xy(-2y)}{2\sqrt{a^2 - x^2 - y^2}}$$

$$\Rightarrow V_y = x\left\{\frac{-x^2 - 2y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} \quad Now\ V_y = 0 \Rightarrow x\left\{\frac{-x^2 - 2y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -x^2 - 2y^2 + a^2 = 0 \Rightarrow x^2 + 2y^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

Solving equations (a) and (b) we get the $x = \dfrac{a}{\sqrt{3}}\ \ and\ \ y = \dfrac{a}{\sqrt{3}}$

Now $V_{xx} = \dfrac{xy(2x^2+3y^2-3a^2)}{(a^2-x^2-y^2)^{\frac{3}{2}}}$ (We obtain this by using quotient rule of differentiation)

$V_{xx}(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}) = -\dfrac{a^2}{\sqrt{3}} \prec 0$

Also we have to calculate $V_{yy} = \dfrac{xy(3x^2+2y^2-3a^2)}{(a^2-x^2-y^2)^{\frac{3}{2}}}$ and $V_{yy}(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}) = -\dfrac{a^2}{\sqrt{3}}$ Also note that $V_{xy}(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}) = -\dfrac{2a}{\sqrt{3}}$ Now as we have the formula for the second order partial derivative is $f_{xx}.f_{yy}-(f_{xy})^2$ and putting the values which we calculated above we note that $f_{xx}(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}).f_{yy}(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}})-(f_{xy}(\frac{a}{\sqrt{3}},\frac{a}{\sqrt{3}}))^2 = +\dfrac{20a^2}{3} \succ 0$ Which shows that the function V has maximum value when $x = \frac{a}{\sqrt{3}}$ *and* $y = \frac{a}{\sqrt{3}}$. So the dimension of the rectangular box are $x = \frac{a}{\sqrt{3}}, y = \frac{a}{\sqrt{3}}$ *and* $z = \frac{a}{\sqrt{3}}$.

Example:

Find the points o the plane x + y + z = 5 in the first octant at which $f(x,y,z) = xy^2z^2$ has maximum value.

Solution:

Since we have $f(x,y,z) = xy^2z^2$ and we are given the plane x + y + z = 5 from this equation we can write x = 5 – y – z . Thus our function “f’ becomes
$f((5-y-z),y,z) = (5-y-z)y^2z^2$ Say this function u(y,z) That is $u(y,z) = (5-y-z)y^2z^2$
Now we have to find out extrema of this function. On simplification we get
$u(y,z) = 5y^2z^2 - y^3z^2 - y^2z^3$
$u_y = 10yz^2 - 3y^2z^2 - 2yz^3$
$\quad = yz^2(10-3y-2z)$
$u_z = 10y^2z - 2y^3z - 3y^2z^2$
$\quad = y^2z(10-2y-3z)$
$u_y = 0, \quad u_z = 0$
$y = 0, \quad z = 0$
$10-3y-2z = 0$
$10-2y-3z = 0$

On solving above equations we get $-10+5z = 0 \Rightarrow z = 2$ and $10-3y-4 = 0 \Rightarrow y = 2$
$u_{yy} = 10z^2 - 6yz^2 - 2z^3$
$u_{zz} = 10y^2 - 2y^3 - 6y^2z$
$u_{yz} = 20yz - 6y^2z - 6yz^2$
at
$y = 2, \quad z = 2$
$u_{yy}(2,2) = 40-48-16 = -24 < 0$
$y_{zz}(2,2) = 40-16-48 = -24$
$u_{yz}(2,2) = 80-48-48 = -16$

$D = u_{yy}\, u_{zz} - (u_{yz})^2$
$= (-24)(-24) - (-16)^2$
$= 576 - 256$
$= 320 > 0$
For $y = 2$ and $z = 2$
We have $x = 5 - 2 - 2 = 1$

Example:

Find all points of the plane $x+y+z=5$ in the first octant at which $f(x,y,z)=xy^2z^2$ has a maximum value.

$f(x,y,z) = xy^2z^2 = xy^2(5 - x - y)^2$

Since $x+y+z = 5$

$f_x = y^2(5-3x-y)(5-x-y)$,
$f_y = 2xy(5-x-2y)(5-x-y)$
Solving $f_x = 0$, $f_y = 0$, we get
$x=1, y=2, z=2 \therefore x+y+z = 5$

$f_{xx}=-y^2(5-3x-y) - 3y^2(5-x-5)$
$f_{xy}=2y(5-x-y)(5-3x-y)-y^2(5-3x-y)$
$\quad - y^2(5 - x - y)$
$f_{yy}=2x(5-x-y)(5-x-2y)-2xy(5-x-2y)$
$\quad - 4xy(5 - x - y)$
$f_{xx}(1, 2, 2) = -24 < 0$
$f_{yy}(1, 2, 2) = -16$
$f_{xy}(1, 2, 2) = -8$
$f_{xx} f_{yy} - (f_{xy})^2 = (-24)(-16)-(-8)^2$
$= 320 > 0$

Hence "f" has maximum value when $x = 1$ and $y = 2$. Thus the points where the function has maximum value is $x = 1, y = 2$ and $z = 2$.

Lecture No -18 Revision of Integration

Example:

Consider the following integral $\int_0^1 (xy + y^2)dx$ Integrating we get

$$\int_0^1 (xy + y^2)dy = x\int_0^1 ydx + \int_0^1 y^2 dx$$

$$= x\left|\frac{y^2}{2}\right|_0^1 + \left|\frac{y^3}{3}\right|_0^1 = y(\frac{1}{2}) + y^2$$

$$\Rightarrow \int_0^1 (xy + y^2)dx = y(\frac{1}{2}) + y^2$$

Example:

Consider the following integral $\int_0^1 (xy + y^2)dy$ Integrating we get

$$\int_0^1 (xy + y^2)dy = x\int_0^1 ydx + \int_0^1 y^2 dx$$

$$= x\left|\frac{y^2}{2}\right|_0^1 + \left|\frac{y^3}{3}\right|_0^1 = x(\frac{1}{2}) + \frac{1}{3}$$

$$\Rightarrow \int_0^1 (xy + y^2)dx = \frac{x}{2} + \frac{1}{3}$$

Double Integral

Symbolically, the double integral of two variables "x" and "y" over the certain region R of the plane is denoted by $\iint_R f(x, y)dxdy$.

Example:

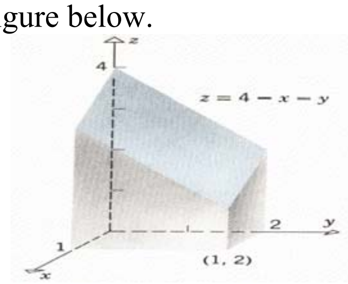
Use a double integral to find out the solid bounded above by the plane Z = 4 – x – y and below by the rectangle $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2\}$

Solution:

We have to find the region "R"out the volume "V" over that is,

$$V = \iint_R (4 - x - y)dA$$

And the solid is shown in the figure below.



$$V = \iint_R (4-x-y)dA = \int_0^2\int_0^1 (4-x-y)dxdy$$

$\int_0^2\int_0^1 (4-x-y)dxdy = \int_0^2 \left| 4x - \frac{x^2}{2} - xy \right|_{x=0}^{x=1} dy$ After putting the upper and lower limits we get

$\int_0^2 \left(\frac{7}{2} - y\right)dy = \left|\frac{7}{2}y - \frac{y^2}{2}\right|_0^2$ again after putting the limits we get the required volume of the solid $V = \iint_R (4-x-y)dA = 5$.

Example:

Evaluate the double integral $\int_0^1\int_0^1 (xy+y^2)dxdy$

Solution:

First we will integrate the given function with respect to "x' and our integral becomes $\int_0^1\int_0^1 (xy+y^2)dydx = \int_0^1 \left(x\left|\frac{y^2}{2}\right|_0^1 + \left|\frac{y^3}{3}\right|_0^1 \right)dy$

and after applying the limits we have,

$$\int_0^1\int_0^1 (xy+y^2)dydx = \int_0^1 \left(\frac{x}{2} + \frac{1}{3}\right)dy \quad integrating\ we\ get$$

$$\int_0^1\int_0^1 (xy+y^2)dydx = \left|\frac{x^2}{4} + \frac{x}{3}\right|_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Iterated or Repeated Integral

The expression $\int_c^d \left[\int_a^b f(x,y)dx\right]dy$ is called iterated or repeated integral. Often the brackets are omitted and this expression is written as

$\int_c^d\int_a^b f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx\right]dy$ Where we have $\int_a^b f(x,y)dx$ yields a function of "y", which is then integrated over the interval $c \le y \le d$.

Similarly $\int_a^b\int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right]dx$ Where we have $\int_c^d f(x,y)dy$ yields a function of "x"which is then integrated the interval $a \le x \le b$.

Example:

Evaluate the integral $\int_0^1\int_0^2 (x+3)dydx$.

Solution:

Here we will first integrate with respect to "y" and get a function of "x" then we will integrate that function with respect to "x" to get the required answer. So

$\int_0^1\int_0^2 (x+3)dydx = \int_0^1 (x+3)\left|y\right|_0^2 dx$ and after putting the limits we get, $\int_0^1 (x+3)\left|y\right|_0^2 dx = \int_0^1 2(x+3)dx = 2\left|\frac{x^2}{2}+3x\right|_0^1$ and the required value of the double integral is $\int_0^1\int_0^2 (x+3)dydx = 2(\frac{1}{2}+3) = 7$.

Now if we change the order of integration so we get $\int_0^2\int_0^1 (x+3)dxdy$ Then we have $\int_0^2\int_0^1 (x+3)dxdy = \int_0^2 \left|(\frac{x^2}{2}+3)\right|_0^1 dy = \int_0^2 \frac{7}{2}dy = \frac{7}{2}\left|y\right|_0^2 = 7$ .Now you note that the value of the integral remain same if we change the order of integration. Actually we have a stronger result which we sate as a theorem.

Theorem:

Let R be the rectangle defined by the inequalities **a < x < b** and **c < y < d.** If f(x, y) is continuous on this rectangle, then $\iint_R f(x,y)dA = \int_c^d\int_a^b f(x,y)dxdy = \int_a^b\int_c^d f(x,y)dxdy$.

Remark:

This powerful theorem enables us to evaluate a double integral over a rectangle by calculating an iterated integral. Moreover the theorem tells us that the **"order of integration in the iterated integral does not matter".**

Example:

Evaluate the integral $\int_0^{\ln 2}\int_0^{\ln 3} e^{x+y}dxdy$

Solution:

First we will integrate the function with respect to "x". Note that we can write e^{x+y} as $e^x.e^y$ So we have, $\int_0^{\ln 2} e^y \left|e^x\right|_0^{\ln 3} dy = \int_0^{\ln 2} e^y(3-1)dy$ Here we use the fact that "e" and "ln" are inverse function of each other. So we have $e^{\ln 3} = 3$ .Thus we get,

$\int_0^{\ln 2} e^y \left|e^x\right|_0^{\ln 3} dy = 2\int_0^{\ln 2} e^y dy = 2\left|e^y\right|_0^{\ln 2} = 2(2-1) = 2$ *is the required answer.*

Example:

Evaluate the integral $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}dydx$ **(Note that in this example we change the order of integration)**

Solution:

First we will integrate the function with respect to "y". Note that we can write e^{x+y} as $e^x.e^y$ So we have, $\int_0^{\ln 3} e^x \left|e^y\right|_0^{\ln 2} dy = \int_0^{\ln 3} e^x(2-1)dy$ Here we use the fact that "e" and "ln" are inverse function of each other. So we have $e^{\ln 2} = 2$ .Thus we get,

$$\int_0^{\ln 3} e^x \left| e^y \right|_0^{\ln 2} dx = \int_0^{\ln 3} e^x dy = \left| e^x \right|_0^{\ln 3} = (3-1) = 2 \ is\ the\ \ required\ \ answer.$$

Note that in both cases our integral has the same value.

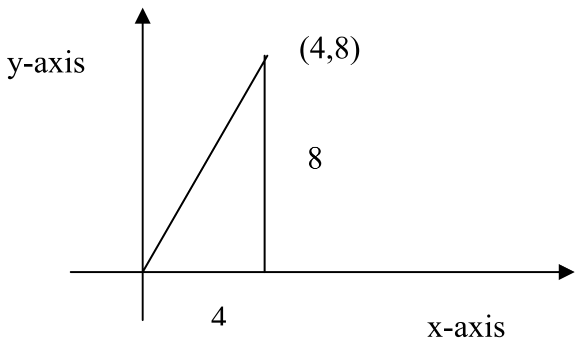
Over view:

Double integrals Page # 854-857

Exercise Set 17.1 (page 857): 1,3,5,7,9,11,13,15,17,19

Lecture No -19 Use Of Integrals

Area as anti-derivatives



$$\int_0^4 2x\,dx = |x^2|_0^4$$
$$= (4)^2 = 16$$

Area of triangle =1/2 base x altitude
$= \frac{1}{2}(4)(8) = 16$

volume as anti-derivative

$$\text{Volume} = \int_0^2\int_0^3 5\,dydx$$
$$= \int_0^2 |5y|_0^3\,dx = \int_0^2 15\,dx$$
$$= |15x|_0^2 = 30$$

$0 \le x \le 2,\quad 0 \le y \le 3,\quad 0 \le z \le 5$
Volume = 2 x 3 x 5 = 30
The following results are analogous to the result of the definite integrals of a function of single variable.

$$\iint_R cf(x,y)\,dxdy$$
$$= c\iint_R f(x,y)\,dxdy \quad (\text{ c a constant })$$

$$\iint_R [f(x,y) + g(x,y)]\,dxdy$$
$$= \iint_R f(x,y)\,dxdy + \iint_R g(x,y)\,dxdy$$
$$\iint_R [f(x,y) - g(x,y)]\,dxdy$$
$$= \iint_R f(x,y)\,dxdy - \iint_R g(x,y)\,dxdy$$

Use double integral to find the volume under the surface $z = 3x^3 + 3x^2y$ and the rectangle $\{(x,y): 1 \le x \le 3,\ 0 \le y \le 2\}$.

$$\text{Volume} = \int_0^2\int_1^3 (3x^3 + 3x^2y)\,dx\,dy$$
$$= \int_0^2 \left|\frac{3x^4}{4} + x^3y\right|_1^3 dy$$
$$= \int_0^2 \left[\frac{3(3)^4}{4} - \frac{3}{4} + (3)^3y - y\right] dy$$
$$= \int_0^2 [60 + 26y]\,dy$$
$$= |60y + 13y^2|_0^2 = 172$$

Use double integral to find the volume of solid in the first octant enclosed by the surface z = x2 and the planes x=2, y =0, y = 3 and z = 0

$$\text{Volume} = \int_0^2 \int_0^3 x^2 \, dy dx$$

$$= \int_0^2 \left| x^2 y \right|_0^3 dx = \int_0^2 [3x^2]\, dx$$

$$= \left| x^3 \right|_0^2 = 8$$

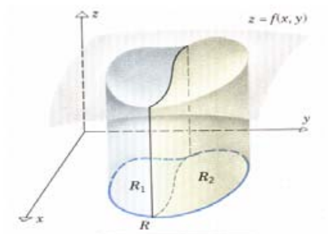
$$\iint_R f(x,y)dA \geq 0 \text{ if } f(x,y) \geq 0 \text{ on } R$$

$$\iint_R f(x,y)dA \geq \iint_R cg(x,y)dA$$

$$\text{if } f(x,y) \geq g(x,y)$$

If $f(x,y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect to subdividing the solid between R and $z= f(x,y)$ into two solids, the sum of whose volumes is the volume of the entire solid

$$\iint_R f(x, y)\, dA = \iint_{R1} f(x, y)\, dA + \iint_{R2} f(x, y) dA$$



The volume of the entire solid is the sum of the volumes of the solids above R_1 and R_2.

The volume of the solid S can also be obtained using cross sections perpendicular to the y -axis.

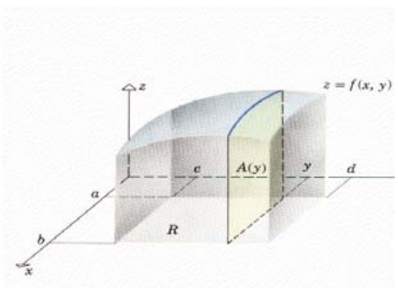
$$\text{Vol (S)} = \int_c^d A(y)\, dy \qquad (1)$$

Where A(y) represents the area of the cross section perpendicular to the y-axis taken at the point y

How to compute cross sectional area

For each fixed y in the interval $c \leq y \leq d$, the function f(x,y) is a function of x alone ,and A(y)may be viewed as the area under the graph of this function along the interval a<x<b, Thus

$$A(y) = \int_a^b f(x, y)\, dx$$



Substiuting this expression in (1) yields.

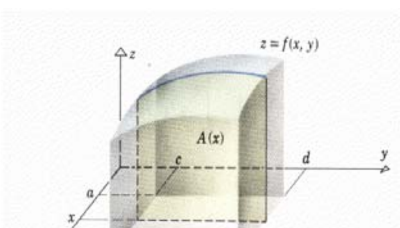
$$\text{Vol (S)} = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

$$= \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Similarly the volume of the side S can also be obtained using sections perpendicular to the x-axis

$$\text{Vol (S)} = \int_a^b A(x) \qquad (3)$$

Where A(x) is the area of the cross section perpendicular to the x-axis taken at the point x.



For each fixed x in the interval a ≤ x ≤ b the function f(x,y) is a function of alone, so that the area A(x) is given by

$$A(x) = \int^{d} f(x,y)dy$$

Substituting this expression in
(3) yields

$$\text{Vol}(S) = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx \qquad (4)$$

From eq (2) and eq (4) we have

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx$$

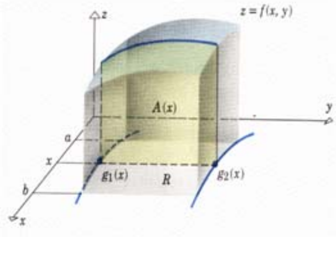
Double integral for non-rectangular region

Type I region is bounded the left and right by the vertical lines x=a and x=b and is bounded below and above by continuous curves $y=g_{1(x)}$ and $y=g_{2(x)}$ where

$$g_{1(x)} \le g_{2(x)} \text{ for } a \le x \le b$$

If R is a type I region on which f(x, y) is continuous, then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_{2(x)}} f(x,y)dydx \quad (1)$$



By the method of cross section, the volume of S is also given by

$$\text{Vol}(S) = \int_a^b A(x)\,dx \qquad (2)$$

where A(x) is the area of the cross section at the fixed point x this cross section area extends from $g_{1(x)}$ to $g_{2(x)}$ in the y-direction,

$$\text{so}\quad, A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

Substituting this in (2) we obtain

$$\text{Vol}(S) = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Since the volume of S is also given by

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II region is bounded below and above by horizontal lines y=c and y=d and is bounded in the left and right by continuous curves $x=h_1(y)$ and $x=h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for for $c \le y \le d$.

. If R is a type II region on which f (x, y) is continuous, then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$$

Similarly, the partial definite integral with respect to $\int_c^d f(x,y)$ is evaluated by holding x fixed and integrating with respect to y.

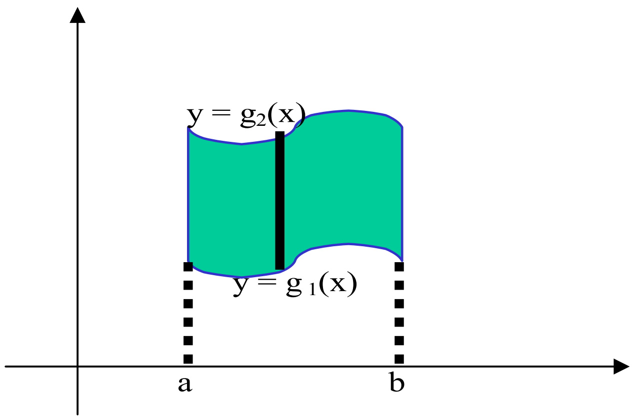
An integral of the form $\int_c^d f(x,y)\,dy$ produces a function of x.

Lecture No -20 Double integral for non-rectangular region

Double integral for non-rectangular region

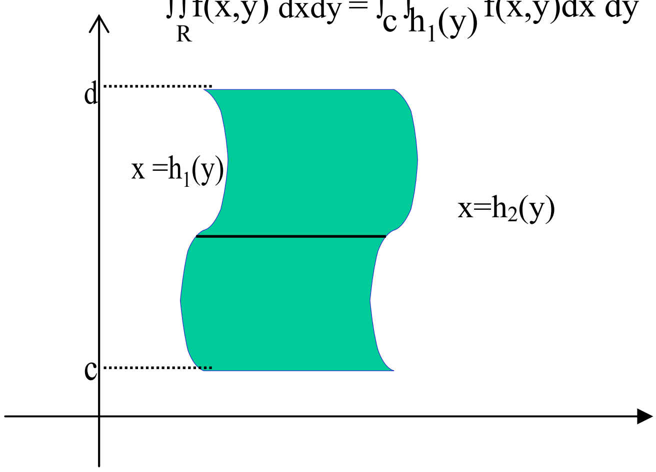
Type I region is bounded the left and right by vertical lines x=a and x=b and is bounded below and above by curves $y=g_1(x)$ and $y=g_2(x)$ where $g_1(x) \le g_2(x)$ for $a \le x \le b$

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$



Type II region is bounded below and above by the horizontal lines y=c and y=d and is bounded on the left and right by the continuous curves $x= h_1(y)$ and $x=h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$

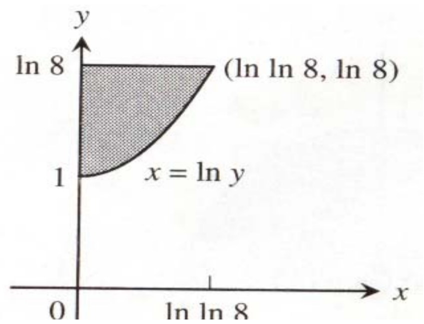
$$\iint_R f(x,y)\, dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\, dy$$



Write double integral of the function *f(x,y)* on the region whose sketch is given

$$\int_1^{\ln 8} \int_0^{\ln y} f(x,y)\, dx$$

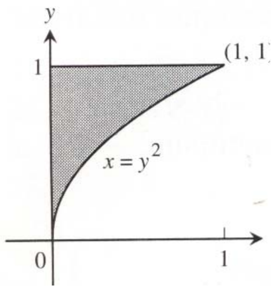
$$\int_0^{\ln(\ln 8)} \int_{e^x}^{\ln 8} f(x,y) \quad y$$



Write double integral of the function *f(x,y)* on the region whose sketch is given

$$\int_0^1 \int_0^{y^2} f(x,y)\, dx\, dy$$

$$\int_0^1 \int_{\sqrt{x}}^1 f(x, y)\, dy\, dx$$



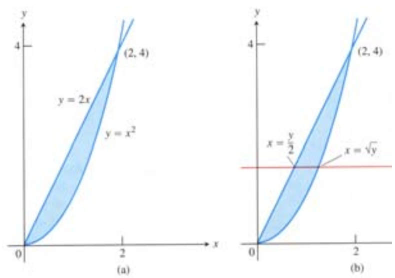
EXAMPLE

Draw the region and evaluate an equivalent integral with the order of integration reversed

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

The region of integration is given by the inequalities $x^2 \leqq y \leqq 2x$ and $0 \leqq x \leqq 2$.

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$



$$= \int_0^4 \left| 2x^2 + 2x \right|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \left[2y + 2\sqrt{y} - \frac{y}{2} - y \right] dy$$

$$= \left| y^2 + \frac{4}{3} y^{3/2} - \frac{y^3}{6} - \frac{y^2}{2} \right|_0^4$$

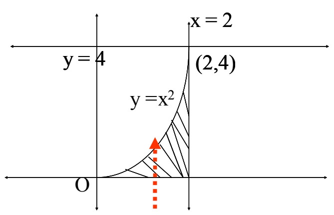
$$= \left[16 + \frac{4}{3}(4)^{3/2} - \frac{(4)^3}{6} - 8 \right]$$

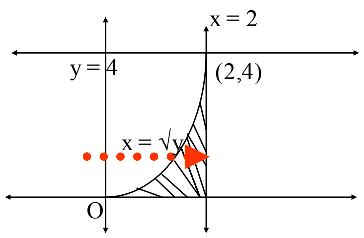
$$= 16 + \frac{4}{3}(8) - \frac{64}{6} - 8 = 8$$

EXAMPLE

Evaluate $I = \int_0^4 \int_{\sqrt{y}}^2 y \cos x^5\, dx\, dy$ The integral is over the region $0 \leq y \leq 4$, $x = \sqrt{y}$ and $x = 2$

$$I = \int_0^2 \int_0^{x^2} y \cos x^5\, dy\, dx$$





$$= \int_0^2 \left[\frac{y^2}{2}\cos x^5\right]_0^{x2} dx$$

$$= \int_0^2 \frac{x^4}{2} \cos x^5 \, dx$$

$$= \frac{1}{10}\int_0^2 \cos x^5 \,.\, (5x^4)\, dx$$

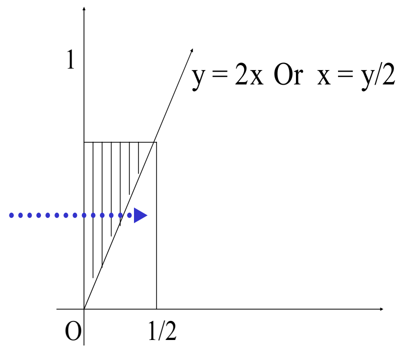
$$= \left[\frac{1}{10}\sin x^5\right]_0^2 = \frac{1}{10} \sin 32$$

Evaluate $I = \int_0^{1/2}\int_{2x}^{1} e^{y^2}\, dy\, dx$

The integral cannot be evaluated I the given order since e^{y^2} has no antiderivative. We shall change the order of integration. The region R which integration is performed is given by $0 \le x \le \frac{1}{2}$, $y = 2x$ and $y = 1$

This region is also enclosed by

$x = 0$, $x = \frac{y}{2}$ and $0 \le y \le 1$



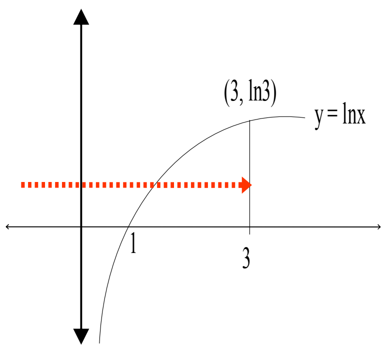
Thus

$$I = \int_0^1\int_0^{y/2} e^{y^2}\, dx\, dy$$

$$= \int_0^1 \frac{y}{2} e^{y^2}\, dy$$

$$= \left[\frac{1}{4} \quad e^{y^2}\right]_0^1 = \frac{1}{4}\,(e-1)$$

$$\int_1^3\int_0^{\ln x} x\, dy\, dx$$



Reversing the order of integration

$$= \int_0^{\ln 3}\int_{e^y}^{3} x\, dx\, dy$$

$$= \int_0^{\ln 3} \left|\frac{x^2}{2}\right|_{e^y}^{3} dy \;=\; \frac{1}{2}\int_0^{\ln 3} [9 - e^{2y}]\, dy$$

$$= \frac{1}{2}\left|9y - \frac{e^{2y}}{2}\right|_{0}^{\ln 3}$$

$$= \frac{1}{2}\left[9\ \ln 3 - \frac{e^{2\ln 3}}{2} + \frac{e^{0}}{2}\right]$$

$$= \frac{1}{2}\left[9\ \ln 3 - \frac{9}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[9\ \ln 3 - 4\right]$$

$$= \frac{9}{2}\ \ln 3 - 2$$

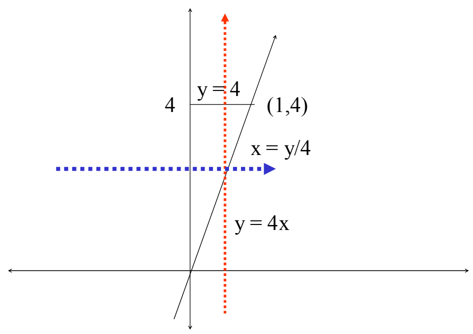
Over view of Lecture # 20

Book Calsulus By Howard Anton
Chapter # !7 Article # 17.2
Page (858-863) Exercise set 17.2
21,22,23,25,27,35,37,38

Lecture No -21 Examples

$$\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx$$

Reversing the order of integration



$$\int_0^4 \int_0^{y/4} e^{-y^2}\, dx\, dy$$

$$= \int_0^4 \left| x\, e^{-y^2} \right|^{y/4} dy \quad = \int_0^4 \frac{y}{4}\, e^{-y^2}\, dy$$

$$= \frac{-1}{8} \int_0^4 e^{-y^2} (-2y)\, dy$$

$$= -\frac{1}{8} \left| e^{-y^2} \right|_0^4 = -\frac{1}{8} \left| e^{-16} - c^0 \right|$$

$$= \frac{1}{8}\left(1 - \frac{1}{e^{16}}\right)$$

Example

Calculate

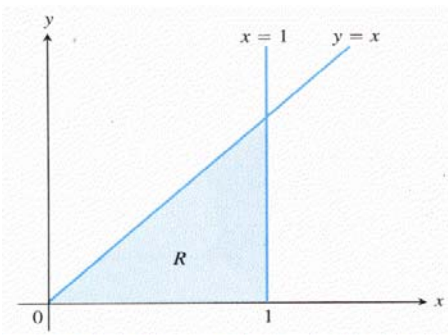
$$\iint_R \frac{\sin x}{x}\, dA.$$

where R is the triangle in the xy- plane bounded by the x-axis ,the line y=x and the line x=1

We integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx$$

$$= \int_0^1 \left(y \frac{\sin x}{x} \Big]_{y=0}^{y=x} \right) dx$$



$$= \int_0^1 \sin x\, dx = -\cos(0) + 1 \approx 0.46$$

$$= \int_0^1 \sin x\, dx = -\cos(0) + 1 \approx 0.46$$

EXAMPLE

$$\int_0^2 \int_{y/2}^1 \mathbf{e}^{\mathbf{x^2}}\, dxdy$$

Since there is no elementary antiderivative of e^{x^2}, the integral

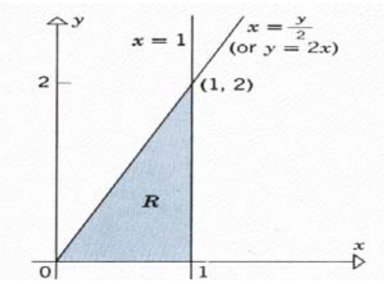
$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dxdy$$

cannot be evaluated by performing the x-integration first.

To evaualte this integral , we express is as an equivalent iterated integral with the order if integration reversed . For the inside integration, y is fixed and x varies from he line $x = y/2$ to the line $x = 1$. For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dxdy$$



By changing the order of integration we get,

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dxdy = \int_0^1 \int_0^{2x} e^{x^2}\, dydx$$

$$= \int_0^1 [e^{x^2} y]_{y=0}^{2x}\, dx$$

$$= \int_0^1 2xe^{x^2}\, dx$$

$$= e^{x^2}\Big|_0^1 = e - 1$$

EXAMPLE

Use a double integral to find the volume of the solid that is bounded above by the palne Z=4-x-y and below by the rectangle $R = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 2\}$

$$V=\iint_R (4-x-y)\,dA$$

$$=\int_0^2\int_0^1 (4-x-y)\,dx\,dy$$

$$=\int_0^2\left[4x-\frac{x^2}{2}-xy\right]_{x=0}^{1} dy$$

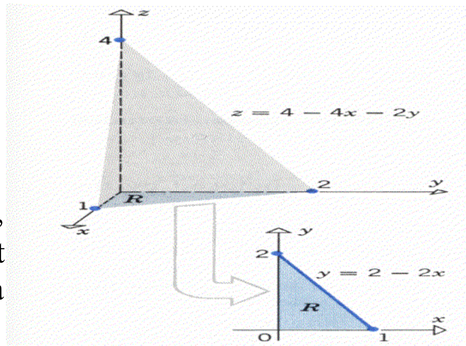
$$=\int_0^2\left(\frac{7}{2}-y\right)dy=\left[\frac{7}{2}y-\frac{y^2}{2}\right]_0^2=5$$

EXAMPLE

Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane z=4-4x-2y The tetrahedron is bounded above by the plane.

$$z=4-4x-2y \quad \text{------------------(1)}$$

and below by the triangular region R



Thus, the volume is given by

$$V = \iint_R (4-4x-2y)\,dA$$

The region R is bounded by the x-axis, the y-axis, and the line y = 2 − 2x [set z = 0 in (1)], so that treating R as a type I region yields.

$$V=\iint_R (4-4x-2y)\,dA$$

$$=\int_0^1\int_0^{2-2x} (4-4x-2y)\,dy\,dx$$

$$=\int_0^1 [4y-4xy-y^2]_{y=0}^{2-2x}\,dx$$

$$=\int_0^1 (4-8x+4x^2)dx$$

$$=\frac{4}{3}$$

Find the volume of the solid bounded by the cylinder x2+y2 = 4 and the planes y + z = 4 and z = 0.

The solid is bounded above by the plane z = 4 − y and below by the region R within the circle $x^2 + y^2 = 4$.

The volume is given by

$$V = \iint_R (4-y)\,dA$$

Treating R as a type I region we obtain

$$V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

$$=\int_{-2}^{2}\left[4y-\frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$=\int_{-2}^{2} 8\sqrt{4-x^2}\,dx$$

$$= 8\left| \frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right|_{-2}^{2}$$

$$= 8\left| 2\sin^{-1}(1) - 2\sin^{-1}(-1) \right|$$

$$= 8[2(\frac{\pi}{2}) + 2(\frac{\pi}{2})]$$

$$= 8(2\pi) = 16\pi$$

EXAMPLE

Use double integral to find the volume of the solid that is bounded above by the paraboiled $z=9x^2 + y^2$,below by the plane $z=0$ and laterally by the planes

$x = 0,\quad y = 0,\quad x = 3,\quad y = 2$

$$\text{Volume} = \int_0^3\int_0^2 (9x^2 + y^2)\, dy\, dx$$

$$= \int_0^3 \left[9x^2y + \frac{y^3}{3} \right]_0^2 dx$$

$$= \int_0^3 \left(18x^2 + \frac{8}{3} \right) dx$$

$$= \left| 6x^3 + \frac{8}{3}x \right|_0^3$$

$$= 6\,(27) + 8$$

$$= 170$$

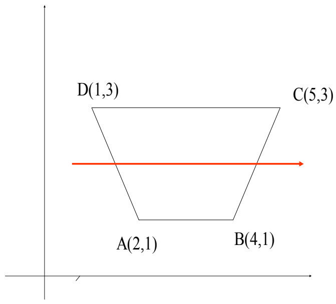
Lecture No -22 Examples

EXAMPLE

$$\iint_R xy \, dA$$

R is the region bounded by the Trapezium with the vertices (1, 3) (5, 3) (2, 1) (4, 1)

Slope of AD $= \dfrac{3-1}{1-2} = -2$



Equation of line AD

$y - 1 = -2(x-2)$

$y - 1 = -2x + 4$

$-2x = y - 5$

$x = -\dfrac{y-5}{2}$

Slope of BC $= \dfrac{3-1}{5-4} = 2$

Equation of line BC

$y - 1 = 2(x-4) \Rightarrow y - 1 = 2x - 8 \Rightarrow 2x = y + 7 \Rightarrow x = \dfrac{y+7}{2}$

$$\int_1^3 \int_{-(y-5)/2}^{(y+7)/2} xy \, dx\, dy = \int_1^3 \left| x^2 \right|_{-(y-5)/2}^{(y+7)/2} y \, dy = \int_1^3 (3y^2 + 3y)\, dy$$

$$= \left| y^3 + \frac{3y^2}{2} \right|_1^3 = (3)^3 + \frac{3(3)^2}{2} - 1 - \frac{3}{2} = 38$$

EXAMPLE

Use double integral to find the volume of the wedge cut from the cylinder $4x^2+y^2=9$ by the plane z=0 and z=y+3

Solution:

Since we can write $4x^2 + y^2 = 9$ as $\dfrac{x^2}{\left(\frac{3}{2}\right)^2} + \dfrac{y^2}{9} = 1$ this is equation of ellipse.

Now the Lower and upper limits for "x" are $x = \dfrac{-\sqrt{9-y^2}}{2}$ *and* $x = \dfrac{\sqrt{9-y^2}}{2}$

And upper and lower limits for "x" are -3 and 3 respectively. So the required volume is given by $\displaystyle\int_{-3}^{3} \int_{\frac{-\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} (y+3)\,dx\,dy$

$$= \int_{-3}^{3} \left| xy + 3x \right|_{-\frac{\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} dy$$

$$= \int_{-3}^{3} \left[y\left(\frac{\sqrt{9-y^2}}{2}\right) + 3\left(\frac{\sqrt{9-y^2}}{2}\right) + y\left(\frac{\sqrt{9-y^2}}{2}\right) + 3\left(\frac{\sqrt{9-y^2}}{2}\right) \right] dy$$

$$= \int_{-3}^{3} \left[y\sqrt{9-y^2} + 3\sqrt{9-y^2} \right] dy$$

$$= -\frac{1}{2}\int_{-3}^{3} \sqrt{9-y^2}\,(-2y)\,dy + 3\int_{-3}^{3} \sqrt{9-y^2}\,dy$$

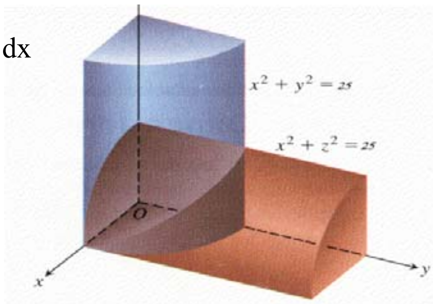
$$= -\frac{1}{2}.\frac{2}{3}\left| (9-y^2)^{\frac{3}{2}} \right|_{-3}^{3} + 3\int_{-3}^{3} \sqrt{9-y^2}\,dy$$

$$= -\frac{1}{3}.[0] + 3\int_{-3}^{3} \sqrt{9-y^2}\,dy = 3\left| \frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\sin^{-1}\left(\frac{y}{2}\right) \right|_{-3}^{3} = \frac{27\pi}{2}$$

EXAMPLE

Use double integral to find the volume of solid common to the cylinders $x^2 + y^2 = 25$ and $x^2 + z^2 = 25$

$$\text{Volume} = 8\int_0^5 \int_0^{\sqrt{25-x^2}} \sqrt{25-x^2}\ dy\,dx$$



$$= 8\int_0^5 \sqrt{25-x^2}\ \left| y \right|_0^{\sqrt{25-x^2}} dx$$

$$= 8\int_0^5 (25-x^2)\,dx$$

$$= 8\left| 25x - \frac{x^3}{3} \right|_0^5 = 8\left(125 - \frac{125}{3}\right) = 8\left(\frac{375-125}{3}\right) = 8\left(\frac{250}{3}\right) = \frac{2000}{3}$$

AREA CALCUALTED AS A DOUBLE INTEGRAL

$$V = \iint_R 1\ dA = \iint_R dA \qquad (1)$$

However, the solid has congruent cross sections taken parallel to the xy-plan so that

V = area of base x height=area of R.1 = area of R

Combining this with (1) yields the area formula

$$\text{area of R} = \iint_R dA \qquad (2)$$

EXAMPLE

Use a double integral to find the area of the region R enclosed between the parabola

$y = ½\, x^2$ and the line $y=2x$

$$\text{area of R} = \iint_R dA$$

$$= \int_0^4 \int_{x^2/2}^{2x} dy\,dx$$

$$= \int_0^4 \left[y\right]_{y=x^2/2}^{2x} dx = \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

Treating R as type IIyields.

$$\text{area of } R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\, dy$$

$$= \int_0^8 \left[x\right]_{y=x^2/2}^{\sqrt{2x}} dx = \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3} y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}$$

EXAMPLE

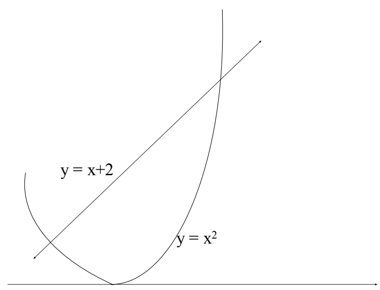
Find the area of the region R enclosed by the parabola $y=x^2$ and the line $y=x+2$

$$\int_{-1}^{2} \int_{y=x^2}^{y=x+2} dy\, dx$$

$$= \int_{-1}^{2} \left[y\right]_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} [x + 2 - x^2] dx$$

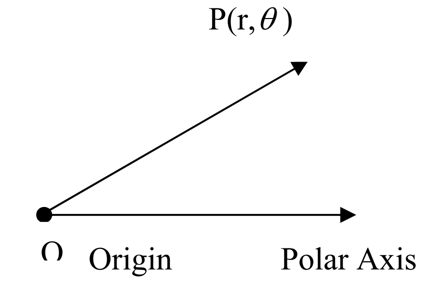
$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$



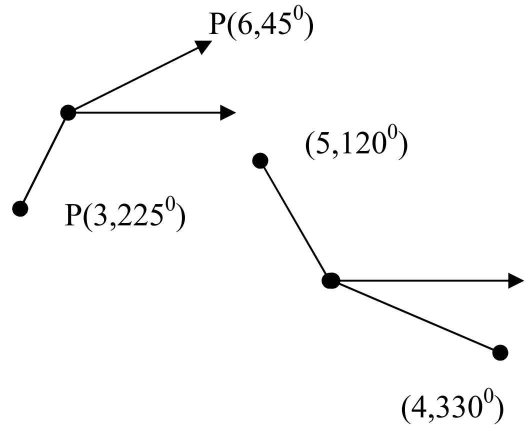
Lecture No - 23 Polar Coordinate Systems

POLAR COORDIANTE SYSTEMS

To form a polar coordinate system in a plane, we pick a fixed point O, called the origin or pole, and using the origin as an endpoint we construct a ray, called the polar axis. After selecting a unit of measurement, we may associate with any point P in the plane a pair of polar coordinates (r, θ), where r is the distance from P to the origin and θ measures the angle from the polar axis to the line segment OP.



The number r is called the radial distance of P and θ is called a polar angle of P. In the points (6, 45°), (3, 225°), (5, 120°), and (4, 330°) are plotted in polar coordinate systems.

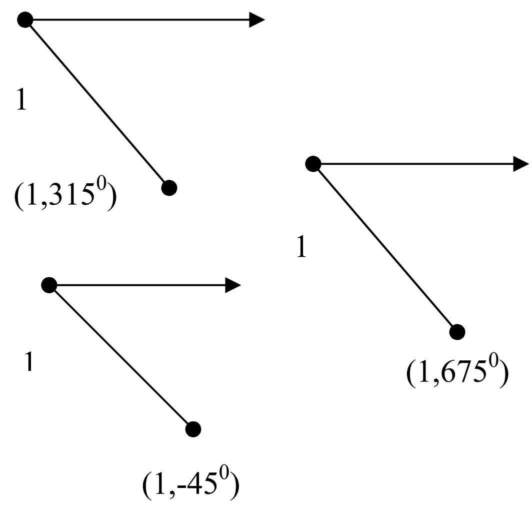


THE POLAR COORDINATES OF A POINT ARE NOT UNIQUE.

For example, the polar coordinates (1, 315°), (1, – 45°), and (1, 675°) all represent the same point

In general, if a point P has polar co-ordinate (r, θ), then for any integer n=0,1,2,3,……. ($r,\theta+n.360^0$) and ($r,\theta+n.360^0$) are also polar co-ordinates of p

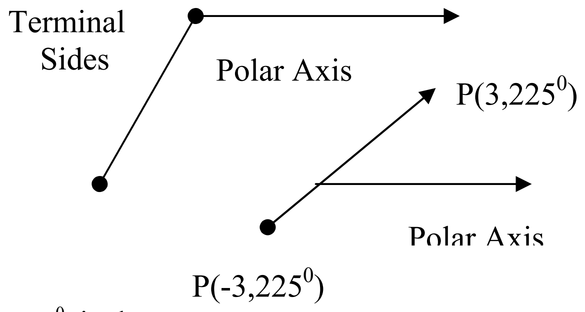
In the case where P is the origin, the line line seg
Because there is no clearly defined polar angle ir
Polar angle θ may be used. Thus, for every θ ma
(0, θ) is the origin.



NEGATIVE VALUES OF R

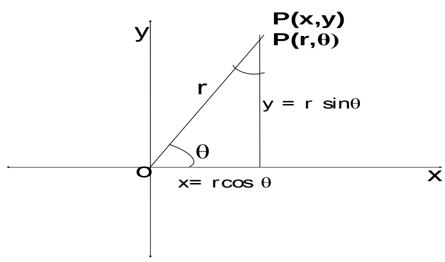
When we start graphing curves in polar coordinates, it will be desirable to allow negative values for r. This will require a special definition. For motivation, consider the point P with polar coordinates ($3, 225^0$) We can reach this point by rotating the polar axis 225^0 and then moving forward from the origin 3 units along the terminal side of the angle. On the other hand, we can also reach the point P by rotating the polar axis 45^0 and then moving backward 3 units from the origin along the extension of the terminal side of the angle

This suggests that the point (3, 225 might also be denoted by (-3, 45 with the minus sign serving to indicate that the point is on the extension of the angle's terminal side rather than on the terminal side itself.

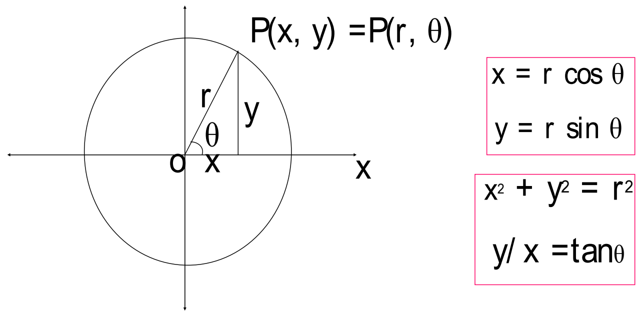


Since the terminal side of the angle $\theta + 180^0$ is the extension of the terminal side other angle θ,We shall define.
(-r, θ) and (r, $\theta + 180^0$) to be polar coordinates for the same point .
With r=3 and θ =45^0 in (2) if follows that (-3, 45^0) and (-3, 225^0) represent the same point.

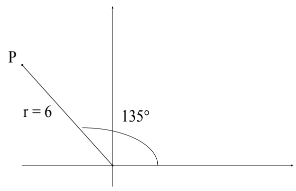
RELATION BETWEEN POLAR AND RECTANGULAR COORDINATES



CONVERSION FORMULA FROM POLAR TO CARTESIAN COORDINATES AND VICE VERSA



Example
Find the rectangular coordinates of the point P whose polar co-ordinates are (6,135^0)
Solution:
Substituting the polar coordinates
r = 6 and θ=135^0 in x = cosθ and y=sinθ yeilds



$x = 6 \cos 135° = 6 (-\sqrt{2}/2) = -3\sqrt{2}$
$y = 6 \sin 135° = 6 (\sqrt{2}/2) = 3\sqrt{2}$
Thus, the rectangular coordinates of the point P are $(-3\sqrt{2}, 3\sqrt{2})$

Example:

Find polar coordinates of the point P whose rectangular coordinates are $(-2, 2\sqrt{3})$

Solution:

We will find polar coordinates (r, θ) of P such that r >0 and $0 \le \theta \le 2\pi$.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

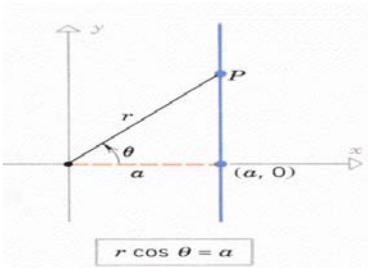
$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

From this we have (-2,2$\sqrt{3}$) lies in the second quadrant of P. All other Polar co-ordinates of P have the form

$$(4, \frac{2\pi}{3} + 2n\pi) \quad or \quad (-4, \frac{5\pi}{3} + 2n\pi) \quad , \quad Where\ n\ is\ integer$$

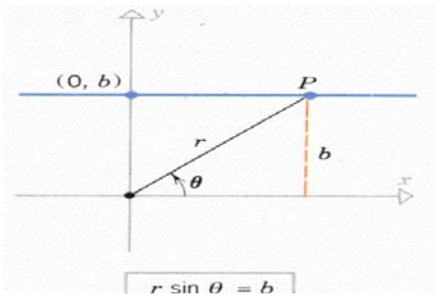
LINES IN POLAR COORDIANTES

A line perpendicular to the x-axis and passing through the point with xy co-ordinates with (a,0) has the equation x=a . To express this equation in polar co-ordinates we substitute x = r cos θ $\Rightarrow$ a = rcos θ --------(1)



This result makes sense geometrically since each point P (r, θ) on this line will yield the value a for r cos θ.

A line parallel to the x-axis that meets the y-axis it the point with xy-coordinates (0, b) has the equation y = b.



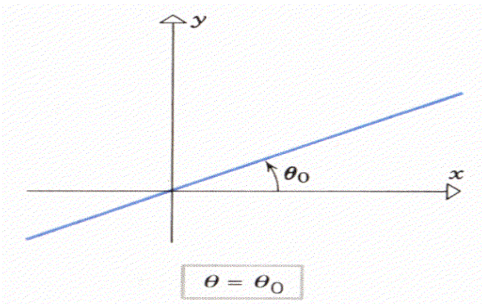
Substituting y = r sin θ yields.

r sin θ = b (2)

as the polar equation of this line. This makes sense geometrically since each point P (r, θ) on this line will yield the value b for r sin θ

For Any constant θ_0, the equation

$\theta = \theta_0$ (3)



is satisfied by the coordinates of all points of the form P (r, θ_0), regardless of the value of r. Thus, the equation represents the line through the origin making an angle of θ_0 (radians) with the polar axis.

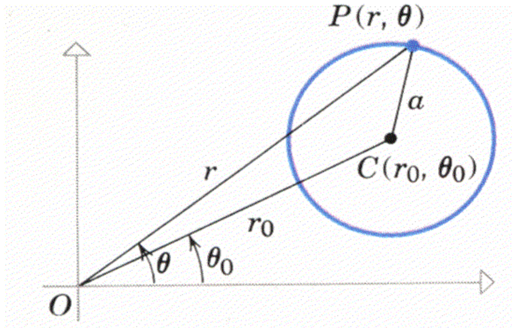
By substitution x = rcosθ and y = rsinθ in the equation Ax +By +C = 0 . We obtain the general polar form of the line,

$$r\,(A\cos\theta + B\sin\theta) + C = 0$$

CIRCLES IN POLAR COORDINATES

Let us try to find the polar equation of a circle whose radius is a and whose center has polar coordinates (r_0, θ_0). If we let $P(r, \theta)$ be an arbitrary point on the circle, and if we apply the law of cosines to the triangle OCP we obtain

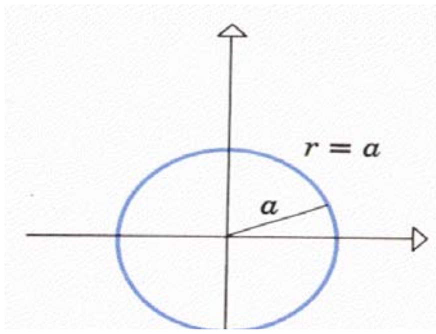
$$r^2 - 2rr_0 \cos(\theta - \theta_0) + r_0^2 = a^2 \qquad (1)$$



SOME SPECIAL CASES OF EQUATION OF CIRCLE IN POLAR COORDINATES

A circle of radius a, centered at the origin, has an especially simple polar equation. If we let $r_0 = 0$ in (1), we obtain $r^2 = a^2$ or, since $a \geq 0$, $r = a$

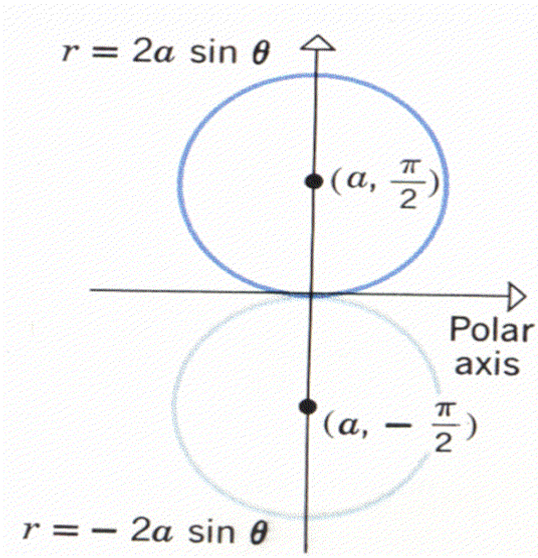
This equation makes sense geometrically since the circle of radius a, centered at the origin, consists of all points $P(r, \theta)$ for which $r = a$, regardless of the value of θ



If a circle of radius a has its center on the x-axis and passes through the origin, then the polar coordinates of the center are either

$(a, 0)$ or (a, π)

depending on whether the center is to the right or left of the origin



Lecture No -24 Sketching

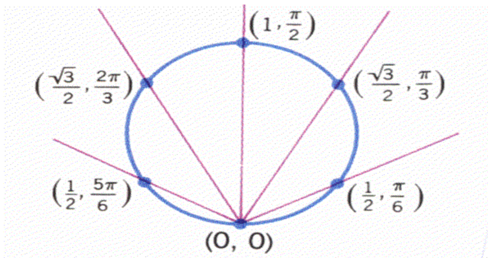
Draw graph of the curve having the equation r = sin θ

By substituting values for θ at increments of $\frac{\pi}{6}(30^0)$ and calculating r , we can construct The following table:

θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
r = sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

θ (radians)	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r = sin θ	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Note that there are 13 pairs listed in Table, but only 6 points plotted in This is because the pairs from θ = π on yield duplicates of the preceding points. For example, (– ½, 7π/6) and (1/2, π/6) represent the same point. The points appear to lie on a circle.



hat this is indeed the case may be seen by expressing the given equation in terms of x and y. We first multiply the given equation through by r to obtain $r^2 = r \sin\theta$ which can be rewritten as

$$x^2 + y^2 = y \quad \text{or} \quad x^2 + y^2 - y = 0$$

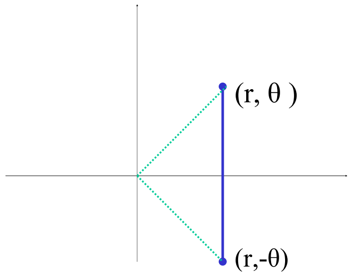
or on completing the square $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$ .This is a circle of radius 1\2 centered at the point (0,1/2) in the xy-plane.

Sketching of Curves in Polar Coordinates

1.SYMMETRY

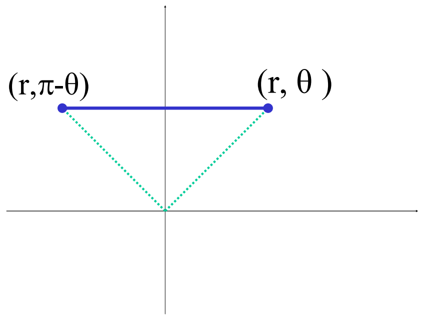
(i) Symmetry about the Initial Line

If the equation of a curve remains unchanged when (r,θ) is replaced by either $(r,-\theta)$ in its equation ,then the curve is symmetric about initial line.



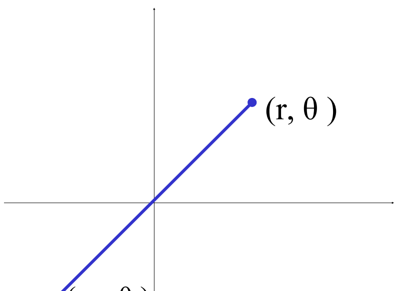
(ii) Symmetry about the y-axis

If when (r, θ) is replaced by either $(r,\pi-\theta)$ in The equation of a curve and an equivalent equation is obtained ,then the curve is symmetric about the line perpendicular to the initial i.e, the y-axis



(ii) Symmetry about the Pole

If the equation of a curve remains unchanged



when either (-r, θ) or is substituted for (r, θ) in its equation ,then the curve is symmetric about the pole. In such a case ,the center of the curve.

2. Position Of The Pole Relative To The Curve

See whether the pole on the curve by putting r=0 in the equation of the curve and solving for θ.

3. Table Of Values

Construct a sufficiently complete table of values. This can be of great help in sketching the graph of a curve.

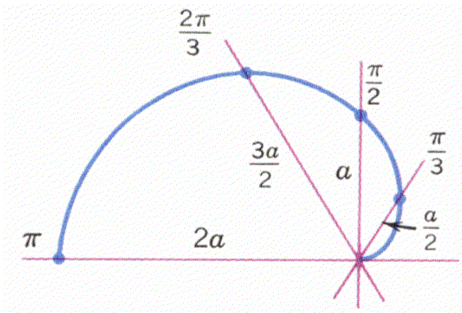
II Position Of The Pole Relative To The Curve.

When r = 0 , $\theta = 0$ .Hence the curve passes through the pole.

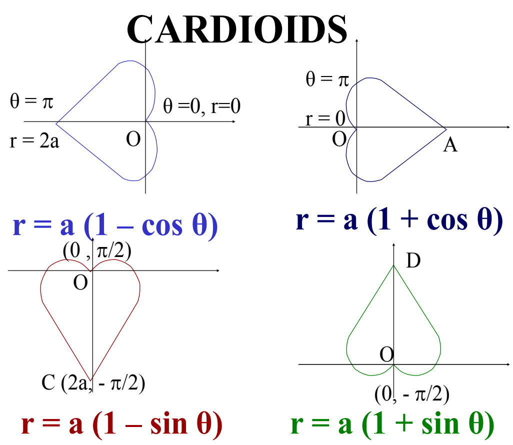
III. Table of Values

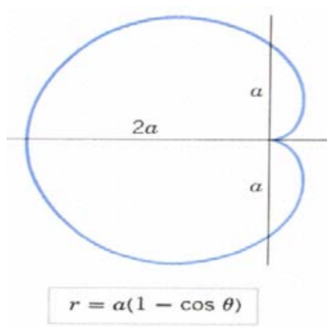
θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$r = a(1-\cos\theta)$	0	a/2	a	3a/2	2a

As θ varies from 0 to π, cos θ decreases steadily from 1 to – 1, and 1 – cos θ increases steadily from 0 to 2. Thus, as θ varies from 0 to π, the value of $r = a(1 - \cos\theta)$ will increase steadily from an initial value of r = 0 to a final value of r = 2a.



On reflecting the curve in about the x-axis, we obtain the curve.



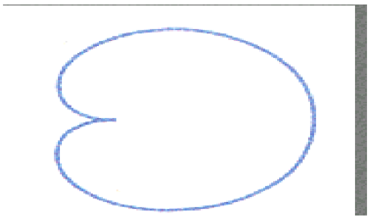


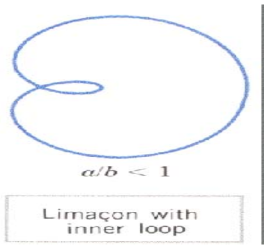
CARDIDOIDS AND LIMACONS

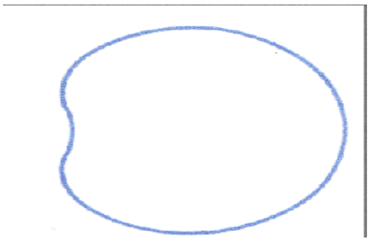
r=a+b sin θ, r = a–bsin θ

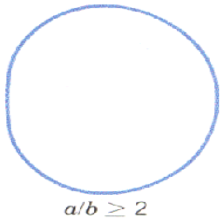
r=a+b cos θ, r = a–bcos θ

The equations of above form produce polar curves called limacons. Because of the heart-shaped appearance of the curve in the case a = b, limacons of this type are called cardioids. The position of the limacon relative to the polar axis depends on whether sin θ or cos θ appears in the equation and whether the + or – occurs.









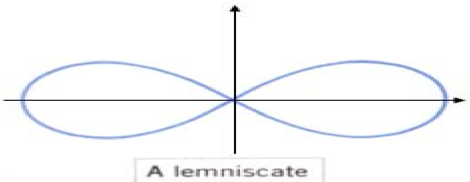
<u>LEMINSCATE</u>

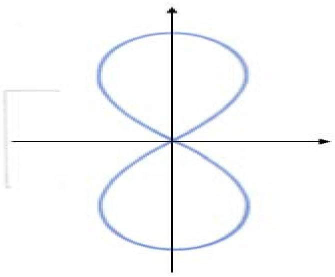
If a > 0, then equation of the form

$$r^2 = a^2 \cos 2\theta, \quad r^2 = -a^2 \cos 2\theta$$

$$r^2 = a^2 \sin 2\theta, \quad r^2 = -a^2 \sin 2\theta$$

represent propeller-shaped curves, called lemiscates (from the Greek word "lemnicos" for a looped ribbon resembling the Fig 8. The lemniscates are centered at the origin, but the position relative to the polar axis depends on the sign preceding the a^2 and whether sin 2θ or cos 2θ appears in the equation.

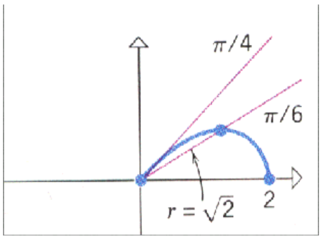


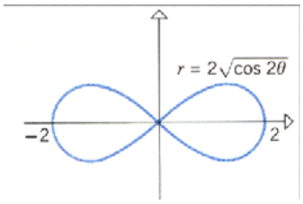


<u>Examp le</u>

$$r^2 = 4 \cos 2\theta$$

The equation represents a lemniscate. The graph is **symmetric** about the **x-axis** and the **y-axis**. Therefore, we can obtain each graph by first sketching the portion of the graph in the range $0 \leq \theta < \pi/2$ and then reflecting that portion about the x- and y-axes.The curve passes through the origin when $\theta = \pi/4$, so the line $\theta = \pi/4$ is tangent to the curve at the origin.As θ varies from 0 to $\pi/4$, the value of $\cos 2\theta$ decreases steadily from 1 to 0, so that rdecreases steadily from 2 to 0.For θ in the range $\pi/4 < \theta < \pi/2$, the quantity $\cos 2\theta$ is negative, so there are no real values of r satisfying first equation. Thus, there are no points on the graph for such θ. The entire graph is obtained by reflecting the curve about the x-axis and then reflecting the resulting curve about the y-axis.



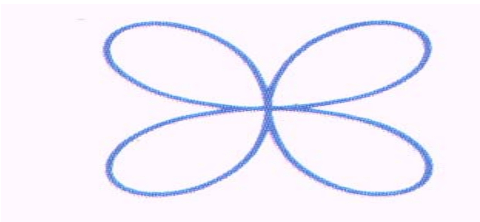


ROSE CURVES

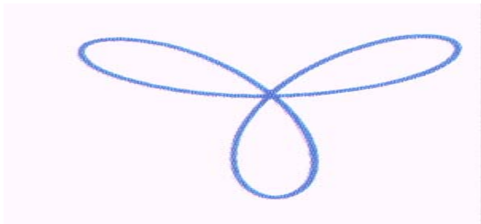
Equations of the form

$r = a \sin n\theta$ and $r = a \cos n\theta$

represent flower-shaped curves called roses. The rose has **n equally spaced petals or loops if n is odd** and **2n equally spaced petals if n is even**



A four-petal rose
($n = 2$)



A three-petal rose
($n = 3$)

The orientation of the rose relative to the polar axis depends on the sign of the constant a and whether $\sin\theta$ or $\cos\theta$ appears in the equation.

SPIRAL

A curve that "winds around the origin" infinitely many times in such a way that r increases (or decreases) steadily as θ increases is called a spiral. The most common example is the spiral of Archimedes, which has an equation of the form.

$$r = a\theta \qquad (\theta \geq 0) \quad \text{or} \quad r = a\theta \qquad (\theta \leq 0)$$

In these equations, θ is in radians and a is positive.

EXAMPLE

Sketch the curve **$r = \theta$ $(\theta \geq 0)$** in polar coordinates.

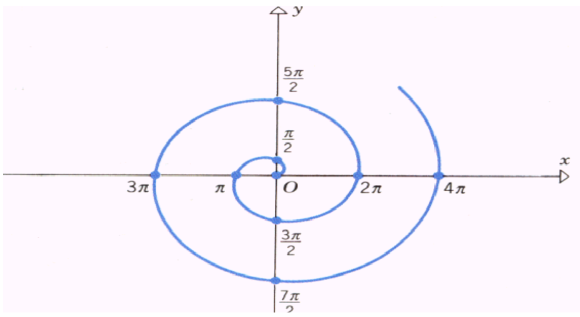
This is an equation of spiral with a = 1; thus, it represents an Archimedean spiral.

Since $r = 0$ when $\theta = 0$, the origin is on the curve and the polar axis is tangent to the spiral.

A reasonably accurate sketch may be obtained by plotting the intersections of the spiral with the x and y axes and noting that r increases steadily as θ increases. The intersections with the x-axis occur when

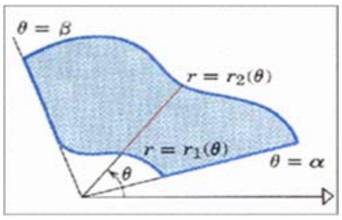
$\theta = 0, \ \pi, \ 2\pi, \ 3\pi, \ldots\ldots$

at which points r has the values
$r = 0,\ \pi,\ 2\pi,\ 3\pi, \ldots..$
and the intersections with the y-axis occur when
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\ldots$$
at which points r has the values
$$r = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\ldots$$
Starting from the origin, the Archimedean spirals $r = \theta$ ($\theta \geq 0$) loops counterclockwise around the origin.



Lecture No -25 Double integrals in polar co-ordinates

Double integrals in which the integrand and the region of integration are expressed in polar coordinates are important for two reasons: First, they arise naturally in many applications, and second, many double integrals in rectangular coordinates are more easily evaluated if they are converted to polar coordinates. The function z = f(r,θ) to be integrated over the region R As shown in the Fig.



INTEGRALS IN POLAR COORDIATES

When we defined the double integral of a function over a region R in the xy-plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant x-values or constant y-values. In polar coordintes, the natural shape is a "polar rectangle" whose sides have constant r and θ- values.

Suppose that a function f (r, θ) is defined over a region R that is bounded by the ray $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = r_1(\theta)$ and $r = r_2(\theta)$. Suppose also that $0 \le r_1(\theta) \le r_2(\theta) \le a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$.

Then the double integral in polar coordinates is given as

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\,dr\,d\theta$$

How to find limits of integration from sketch

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ. This line crosses the boundary of R at most twice. The innermost point of intersection is one the curve $r = r_1(\theta)$ and the outermost point is on the curve $r = r_2(\theta)$. These intersections determine the r-limits of integration.

Step 2. Imagine rotating a ray along the positive x-axis one revolution counterclockwise about the origin. The smallest angle at which this ray intersects the region R is $\theta = \alpha$ and the largest angle is $\theta = \beta$. This yields the θ-limits of the integration.

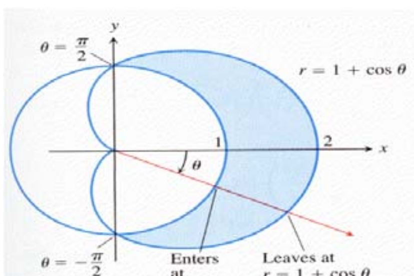
EXAMPLE

Find the limits of integration for integrating f (r, θ) over the region R that lies inside the cardioid r = 1 + cos θ and outside the circle r = 1.

Solution:

Step 1. We sketch the region and label the bounding curves.

Step 2. The r-limits of integration. A typical ray from



The origin enters R where r =1 and leaves where r =1+cos θ.

Step 3. The θ-limits of integration. The rays from the origin that intersect R run from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

The integral is $\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$

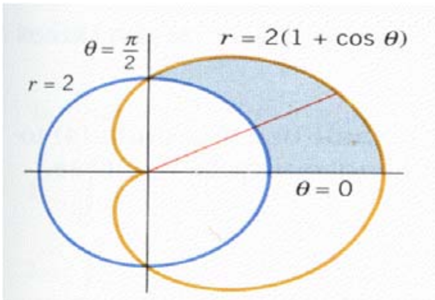
$= 2\int_{0}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\, r dr\, d\theta$

EXAMPLE

Evalaute $\iint_R \sin\theta\, dA$

Where R is the region ion the first quadrant that is outside the circle r = 2 and inside the cardioid r = 2(1+cosθ).

Solution:



$\iint_R \sin\theta\, dA = \int_{0}^{\pi/2}\int_{2}^{2(1+\cos\theta)} (\sin\theta)\, r\, dr\, d\theta$

$= \int_{0}^{\pi/2} \frac{1}{2} r^2 \sin\theta\Big]_{r=2}^{2(1+\cos\theta)} d\theta$

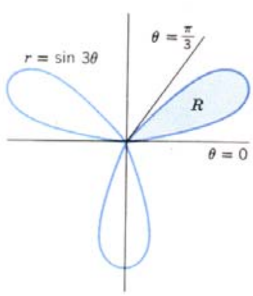
$= 2\int_{0}^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta]\, d\theta$

$= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\pi/2} = \left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$

EXAMPLE

Use a double polar integral to find the area enclosed by the three-petaled rose r = sin 3θ.
We calculate the area of the petal R in the first quadrant and multiply by three.

Solution:



$A = 3\iint_R dA = 3\int_{0}^{\pi/3}\int_{0}^{\sin 3\theta} r\, dr\, d\theta = \int_{0}^{\pi/3} \frac{r^2}{2}\Big|_{0}^{\sin 3\theta} dr\, d\theta$

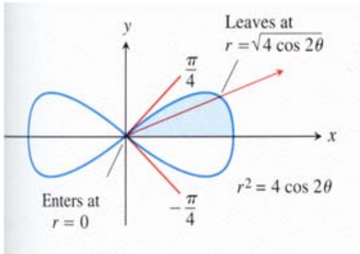
$= \frac{3}{2}\int_{0}^{\pi/3} \sin^2 3\theta\, d\theta \;= \frac{3}{4}\int_{0}^{\pi/3} (1-\cos 6\theta)\, d\theta$

$= \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \left[\frac{3}{4}\theta - \frac{3}{24}\sin 6\theta\right]_0^{\pi/3} = \frac{1}{4}\pi$

EXAMPLE

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$. The total area is four times the first-quadrant portion.

Solution:



$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = A\int_0^{\pi/4}\left[\frac{r2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

CHANGING CARTESIAN INTEGRALS INTO POLAR INTEGRALS

The procedure for changing a Cartesian integral $\iint_R f(x, y)\,dx\,dy$ into a polar integral has two steps.

Step 1. Substiute x = r cos θ and y = r sinθ, and replace dx dy by r dr dθ in the Cartesian integral.

Step 2. Supply polar limits of integration for the boundary of R. The Cartesian integral then becomes

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where G denotes the region of integration in polar coordinates.

Notice that dx dy is not replaced by dr dθ but by r dr dθ.

EXAMPLE

Evalaute the double integral $\int_0^1\int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$ by changing to polar coordinates.

The region of integration is bounded by

$0 \le y \le \sqrt{1 - x^2}$ and $0 \le x \le 1$

$y = \sqrt{1 - x^2}$ is the circle

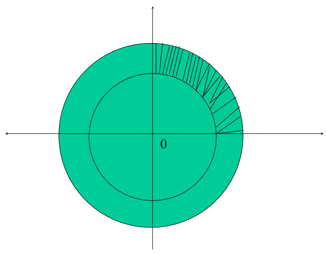
$x^2 + y^2 = 1, \quad r = 1$

On changing into the polar coordinates, the given integral is

$$\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \int_0^{\pi/2}\left|\frac{r^4}{4}\right|_0^1 d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \left|\frac{\theta}{4}\right|_0^{\pi/2} = \frac{\pi}{8}$$

EXAMPLE

Evaluate $I = \int\int_D \frac{dx\,dy}{x^2 + y^2}$ by changing to polar coordinates,



where D is the region in the first quadrant between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $0<a<b$

$$I = \int_0^{\pi/2}\int_a^b \frac{r\,dr\,d\theta}{r^2} = \int_0^{\pi/2} [\ln r]_a^b\,d\theta$$

$$= \int_0^{\pi/2} \ln\left(\frac{b}{a}\right) d\theta = \Big[\theta \ln\left(\frac{b}{a}\right)\Big]_0^{\pi/2} = \frac{\pi}{2} \ln\left(\frac{b}{a}\right).$$

EXAMPLE

Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$ by changing to polar coordinates.

The region of integration is bounded by $0 < y < \sqrt{1-x^2}$ and $0 \le x \le 1$

$y = \sqrt{1-x^2}$ is the circle $x^2+y^2=1$, r = 1

On changing into the polar coordinates, the given integral is

$$\int_0^{\pi/2}\int_0^1 r^3 dr d\theta = \int_0^{\pi/2} \left|\frac{r^4}{4}\right|_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{1}{4}|\theta|_0^{\pi/2} = \frac{1}{4}(\pi/2) = \pi/8$$

Lecture No - 26 EXAMPLES

EXAMPLE

Evalaute $I=\int_0^4\int_0^{\sqrt{4y-y^2}} (x^2+y^2)dx\,dy$ by changing into polar coordinates.

The region of integration is bounded by $0\le x\le\sqrt{4y-y^2}$ and $0 \le y \le 4$

Now $x = \sqrt{4y-y^2}$ is the circle $x^2+y^2-4y=0 \Rightarrow x^2 + y^2 = 4y$.In polar coordinates this takes the form $r^2 = 4r\sin\theta$, $r = 4\sin\theta$

On changing the integral into polar coordinates, we have

$$I = \int_0^{\pi/2}\int_0^{4\sin\theta} r^2 . r\,dr\,d\theta = \int_0^{\pi/2} 64\sin^4\theta\,d\theta = 64.\frac{3.1}{4.2}.\frac{\pi}{2} = 12\pi \quad \text{(using Walli's formula)}$$

EXAMPLE

Evalaute $\iint_R e^{x^2+y^2}\,dy\,dx$.,where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y.

Substituting $x = r\cos\theta$, $y = r\sin\theta$, and replacing dy dx by $r\,dr\,d\theta$ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}dy\,dx = \int_0^{\pi}\int_0^{1} e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta = \int_0^{\pi}\frac{1}{2}(e-1)d\theta = \frac{\pi}{2}(e-1).$$

EXAMPLE

Let R_a be the region bounded by the circle $x^2 + y^2 = a^2$. Define

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \lim_{a\to\infty}\iint_R e^{-(x^2+y^2)}dx\,dy$$

To evaluate this improper integral.

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-(x^2+y^2)\}\,dx\,dy = \lim_{a\to\infty}\iint_{Da}\exp\{-(x^2+y^2)\}\,dx\,dy$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^{a}\exp\{-r^2\}r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\frac{1}{2}(1-\exp\{-a^2\}d\theta = \lim_{a\to\infty}\frac{1}{2}(1-\exp\{-a^2\}\,\theta\Big|_0^{2\pi}$$

$$= \pi - \lim_{a\to\infty}\frac{\pi}{\exp\{-a^2\}} = \pi$$

EXAMPLE

Prove that $\int_0^{\infty} e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}$.

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-(x^2+y^2)\}\,dx\,dy = \int_{-\infty}^{\infty} \exp\{-y^2\}[\int_{-\infty}^{\infty} \exp\{-x^2\}dx]\,dy = [\int_{-\infty}^{\infty} \exp\{-y^2\}dy][\int_{-\infty}^{\infty} \exp\{-x^2\}dx]$$

$$= [\int_{-\infty}^{\infty} \exp\{-t^2\}dt][\int_{-\infty}^{\infty} \exp\{-t^2\}\,dt] = \lim_{a\to\infty}[\int_{-a}^{a} \exp\{-t^2\}\,dt]^2 = 4\lim_{a\to\infty}[\int_0^{a} \exp\{-t^2\}\,dt]^2$$

Hence we have $4\lim_{a\to\infty}[\int_0^{a} \exp\{-t^2\}\,dt]^2 = \lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} \exp\{-(x^2+y^2)\}dx\,dy \quad 4\lim_{a\to\infty}[\int_0^{\infty} \exp\{-t^2\}dt]^2 = \pi$

$$\lim_{a\to\infty}[\int_0^{\infty} \exp\{-t^2\}dt]^2 = \pi/4 \Rightarrow \int_0^{\infty} \exp\{-t^2\}\,dt = \frac{\sqrt{\pi}}{2}$$

THEOREM

Let G be the rectangular box defined by the inequalities

$a \le x \le b, \quad c \le y \le d, \qquad k \le z \le \ell$

If f is continuous on the region G, then $\iiint_G f(x, y, z)\,dV = \int_a^b \int_c^d \int_k^{\ell} f(x, y, z)\,dz\,dy\,dx$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

$$= \int_c^d \int_k^{\ell} \int_a^b f(x, y, z)\,dy\,dz\,dx = \int_c^d \int_a^b \int_k^{\ell} f(x, y, z)\,dy\,dx\,dz = \int_a^b \int_c^d \int_k^{\ell} f(x, y, z)\,dx\,dy\,dz$$

$$= \int_a^b \int_k^{\ell} \int_c^d f(x, y, z)\,dx\,dz\,dy = \int_k^{\ell} \int_a^b \int_c^d f(x, y, z)\,dz\,dx\,dy$$

EXAMPLE

Evalaute the triple integral $\iiint_G 12xy^2z^3\,dV$ over the rectangular box G defined by the inequalities $-1 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 2$.

We first integrate with respect to z, holding x and y fixed, then with respect to y holding x fixed, and finally with respect to x.

$$\iiint_G 12xy^2z^3\,dV = \int_{-1}^{2}\int_0^{3}\int_0^{2} 12xy^2z^3\,dz\,dy\,dx = \int_{-1}^{2}\int_0^{3} [3xy^2z^4]_{z=0}^{2}\,dy\,dx = \int_{-1}^{2}\int_0^{3} 48xy^2\,dy\,dx$$

$$= \int_{-1}^{2} [16xy^3]_{y=0}^{3}\,dx = \int_{-1}^{2} 432x\,dx = 216x^2\Big]_{-1}^{2} = 648$$

EXAMPLE

Evalaute $\iint_R\int (x - 2y + z)\, dx\, dy\, dz$ Region R : $0 \le x , 1,\ 0 \le y \le x^2,\ 0 \le z \le x + y$

$$= \int_0^1\int_0^{x^2}\int_0^{x+y} (x - 2y + z)\, dz\, dy\, dx = \int_0^1\int_0^{x^2} \left|\frac{(x - 2y + z)^2}{2}\right|_0^{x+y} dy\, dx$$

$$= \int_0^1\int_0^{x^2} \left[\frac{(x-2y+x+y)^2}{2} - \frac{(x-2y)^2}{2}\right] dy\, dx = \frac{1}{2}\int_0^1\int_0^{x^2} (3x2 - 3y2)\, dy\, dx = \frac{3}{2}\int_0^1 \left|x^2y - \frac{y^3}{3}\right|_0^{x^2} dx$$

$$= \frac{3}{2}\int_0^1 \left(x4 - \frac{x^6}{z}\right) dx = \frac{3}{2}\left|\frac{x^5}{5} - \frac{x^7}{21}\right|_0^1 = \frac{3}{2}\left[\frac{1}{5} - \frac{1}{21}\right] = \frac{8}{35}$$

Example:

Evalaute $\iiint_S xyzd\, x\, dy\, dz$ Where $S = \{(x,y,z): x^2+y^2+z^2 \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0\}$

S is the sphere $x^2 + y^2 + z^2 = 1$ .Since x, y, z are al +ve so we have to consider only the +ve octant of the sphere.

Now $x^2 + y^2 + z^2 = 1$. So that $z = \sqrt{1 - x^2 - y^2}$

The Projection of the sphere on xy plan is the circle $x^2 + y^2 = 1$.

This circle is covered as y-varies from 0 to $\sqrt{1 - x^2}$ and x varies from 0 to 1.

$$\iiint_R xyz\, dx\, dy\, dz = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx = \int_0^1\int_0^{\sqrt{1-x^2}} xy\left|\frac{z^5}{2}\right|_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} xy\left(\frac{1 - x^2 - y^2}{2}\right) dy\, dx = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} x\,(y - x^2y - y^3)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left(\frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right)\Big|_0^{\sqrt{1-x^2}} dx = \frac{1}{4}\int_0^1 x\left[1-x^2-x^2(1-x^2) - \frac{(1-x^2)^2}{2}\right] dx$$

$$= \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\, dx = \frac{1}{8}\left|\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6}\right|_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

Lecture No -27 Vector Valued Functions

Recall that a function is a rule that assigns to each element in its domain one and only one element in its range. Thus far, we have considered only functions for which the domain and range are sets of real numbers; such functions are called real-valued functions of a real variable or sometimes simply real-valued functions. In this section we shall consider functions for which the domain consists of real numbers and the range consists of vectors in 2-space or 3-space; such functions are called vector-valued functions of a real variable or more simply vector-valued functions. In 2-space such functions can be expressed in the form.

$\mathbf{r}\ (\mathbf{t}) = (\mathbf{x}\ (\mathbf{t}), \mathbf{y}\ (\mathbf{t})) = \mathbf{x}\ (\mathbf{t})\ \mathbf{i} + \mathbf{y}\ (\mathbf{t})\ \mathbf{j}$

and in 3-space in the form

$\mathbf{r}\ (\mathbf{t}) = (\mathbf{x}\ (\mathbf{t}), \mathbf{y}\ (\mathbf{t}), \mathbf{z}\ (\mathbf{t}))\mathbf{i}+\mathbf{y}(\mathbf{t})\mathbf{j}+\mathbf{z}\ (\mathbf{t})\mathbf{k}$

where x(t), y(t), and z(t) are real-valued functions of the real variable t. These real-valued functions are called the component functions or components of **r**. As a matter of notation, we shall denote vector-valued functions with boldface type [**f(t), g(t)**, and **r** (t) and real-valued functions, as usual, with lightface italic type [*f(t), g(t), and r(t)*].

EXAMPLE

$\mathbf{r}\ (t) = (\ln t)\ \mathbf{i} + \sqrt{t^2 + 2}\ \mathbf{j} + (\cos t\pi)\mathbf{k}$

then the component functions are $x(t)=\ln t, y(t)=\sqrt{t^2 + 2}$, and $z\ (t) = \cos t\pi$

The vector that r(t) associates with t = 1 is $r(1)=(\ln 1)\ \mathbf{i}+\sqrt{3}\mathbf{j} +(\cos \pi)\ \mathbf{k}=\sqrt{3}\mathbf{j}- \mathbf{k}$

The function **r** is undefined if $t \leq 0$ because ln *t* is undefined for such *t*.

If the domain of a vector-valued function is not stated explicitly, then it is understood to consist of all real numbers for which every component is defined and yields a real value. This is called the natural domain of the function. Thus the natural domain of a vector-valued function is the intersection of the natural domains of its components.

PARAMETRIC EQUATIONS IN VECTOR FORM

Vector-valued functions can be used to express parametric equations in 2-space or 3-space in a compact form.

For example, consider the parametric equations $x = x(t),\ y = y\ (t)$

Because two vectors are equivalent if and only if their corresponding components are equal, this pair of equations can be replaced by the single vector equation.

$x = x(t),\ y = y\ (t)$

$x\ \mathbf{i} + y\ \mathbf{j} = x\ (t)\mathbf{i} + y\ (t)\ \mathbf{j}$

Similarly, in 3-space the three parametric equations

$x = x(t),\ y = y\ (t),\ z = z\ (t)$

can be replaced by the single vector equation

$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

if we let $r = x\ \mathbf{i} +y\ \mathbf{j}$ and $\mathbf{r}(t) = x(t)\ \mathbf{i} + y(t)\ \mathbf{j}$ in 2-sapce

and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}(t) = x(t)\ \mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

in 3-space, then both (2) and (4) can be written as $\mathbf{r} = \mathbf{r}(t)$

which is the vector form of the parametric equations in (1) and (3). Conversely, every vector equation of form (5) can be rewritten as parametric equations by equating components on the two sides.

EXAMPLE

Express the given parametric equations as a single vector equation.

(a) $x = t^2$, $y = 3t$

(b) $x = \cos t$, $y = \sin t$, $z = t$

(a) Using the two sides of the equations as components of a vector yields.

$$x\,\mathbf{i} + y\,\mathbf{j} = t^2\,\mathbf{i} + 3t\,\mathbf{j}$$

(b) Proceeding as in part (a) yields

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

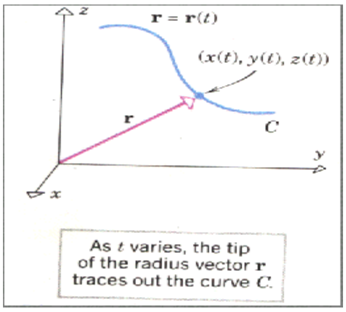
EXAMPLE

Find parametric equations that correspond to the vector equation

$$\mathbf{x\,i + y\,j + z\,k = (t^3 + l)\,i + 3\,j + e^t\,k}$$

Equating corresponding components yields.

$x = t^3 + 1$, $y = 3$, $z = e^t$



GRAPHS OF VECOR-VALUED FUNCTOINS

One method for interpreting a vector-valued function r(t) in 2-space or 3-space geometrically is to position the vector r = r (t) with its initial point at the origin, and let C be the curve generated by the tip of the vector r as the parameter t varies

The vector r, when positioned in this way, is called the radius vector or position vector of C, and C is called the graph of the function r (t) or, equivalently, the graph of the equation r = r (t). The vector equation r = r (t) is equivalent to a set of parametric equations, so C is also called the graph of these parametric equations.

EXAMPLE

Sketch the graph of the vector-valued function r(t) = (cos t)i + (sin t)j, $0 \le t \le 2\pi$

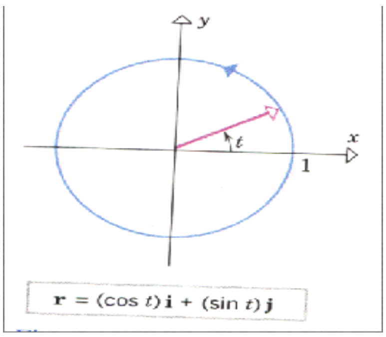
The graph of r (t) is the graph of the vector equation

$x\mathbf{i}+y\mathbf{j} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

or equivalently, it is the graph of the parametric equations

$x = \cos t$, $y = \sin t$ $(0 \le t \le 2\pi)$

This is a circle of radius 1 that is centered at the origin with the direction of increasing t counterclockwise. The graph and a radius vector are shown in Fig.



EXAMPLE

Sketch the graph of the vector-valued function r(t) = (cos t)i+(sin t)j+2k, $0 \le t \le 2\pi$

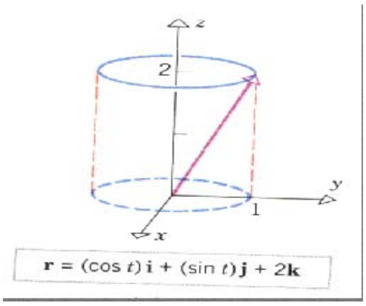
The graph of **r**(t) is the graph of the vector equation

x**i**+y**j**+z**k**=(cost)**i**+(sint)**j**+2k, $0 \le t \le 2\pi$

or, equivalently, it is the graph o the parametric equations

x=cos t, y = sin t, z=2 $(0 \le t \le 2\pi)$

From the last equation, the tip o the radius vecor traces a curve in the plane z = 2, and from the first two equations and the preceding example, the curve is a circle of radius 1 centered on the z-axis and traced counterclockwise looking down the z-axis. The graph and a radius vector are shown in Fig.



EXAMPLE

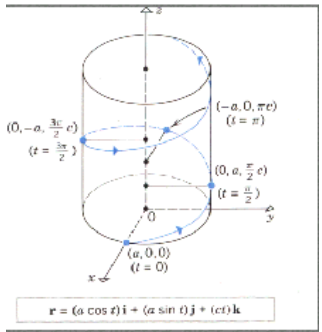
Sketch the graph of the vector-valued function r(t) = (a cos t)i + (a sin t)j + (ct)k where a and c are positive constant.

The graph of **r**(t) is the graph of the parametric equations.

x = a cos t, y = a sin t, z = ct

As the parameter t increases, the value of z = ct also increases, so the point (x, y, z) moves upward. However, as t increases, the point (x, y, z) also moves in a path directly over the circle. x = a cos t, y = a sin t in the xy-plane. The combination of these upward and circular motions produces a corkscrew-shaped curve that wraps around a right-circular cylinder of radius a centered on the z-axis.

This curve is called a circular helix.



EXAMPLE

Describe the graph of the vector equation r = (– 2 + t)i + 3tj + (5 – 4t)k

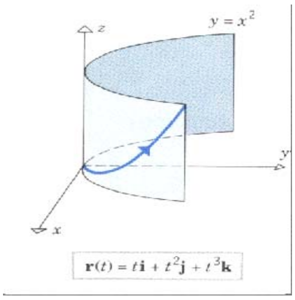
The corresponding parametric equations are x = – 2 + t, y = 3t, z = 5 – 4t

The graph is the line in 3-space that passes through the point (– 2, 0, 5) and is parallel to the vector **i** + 3**j** – 4**k**.

EXAMPLE

The graph of the vector-valued function r (t) = t **i** + t^2 **j** + t^3 **k** is called a twisted cubic. Show that this curve lies o the parabolic cylinder $y = x^2$, and sketch the graph for $t \ge 0$

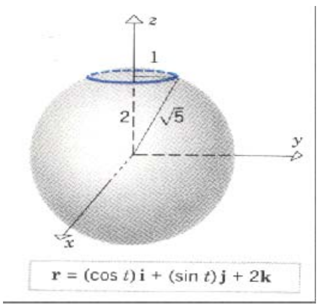
The corresponding parametric equations are $x = t$, $y = t^2$, $z = t^3$
Eliminating the parameter t in the equations for x and y yields $y = x^2$, so the curve lies on the parabolic cylinder with this equation. The curve starts at the origin for t = 0; as t increases, so do x, y, and z, so the curve is traced in the upward direction, moving away from the origin along the cylinder.



GRAPHS OF CONSTANT VECOR-VALUED FUNCTIONS
If c is a constant vector in the sense that it does not depend on a parameter, then the graph of **r** = **c** is a single point since the radius vector remains fixed with its tip at **c**.
If **c** = $x_0\mathbf{i} + y_0\mathbf{j}$ (in 2-space), then the graph is the point (x_0, y_0), and if $c = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ (in 3-space), then the graph is the point (x_0, y_0, z_0).

EXAMPLE
The graph of the equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ is the point (2, 3, − 1) in 3-space.
If r(t) is a vector-valued function, then for each value of the parameter t, the expression ||r(t)|| is a real-valued function of t because the norm (or length of r(t) is a real number.
For example,
If $\mathbf{r}(t) = t\,\mathbf{i} + (t-1)\,\mathbf{j}$
Then $\|r(t)\| = \sqrt{t^2 + (t-1)^2}$ which is a real-valued function of t.

EXAMPLE
The graph of $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}$, $0 \le t \le 2\pi$
is a circle of radius 1 centered on the z-axis and lying in the plane z = 2. This circle lies on the surface of a sphere of radius $\sqrt{5}$ because for each value of t
$\|r(t)\| = \sqrt{\cos^2 t + \sin^2 t + t} = \sqrt{1+4} = \sqrt{5}$
which shows that each point on the circle is a distance of $\sqrt{5}$ units from the origin.



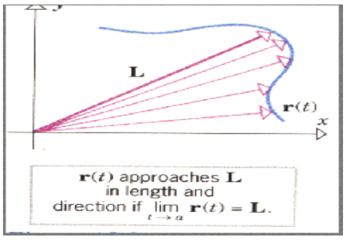
Lecture No -28 Limits of Vector Valued Functions

The limit of a vector-valued functions is defined to be the vector that results by taking the limit of each component. Thus, for a function r(t) = x (t)**i** + y (t)**j** in 2-space we define.

$$\lim_{t\to\alpha} r(t) = (\lim_{t\to\alpha} x(t))\mathbf{i} + (\lim_{t\to\alpha} y(t))\mathbf{j}$$

and for a function r(t) = x(t)**i** + y(t)**j** + z(t)**k** in 3-space we define.

$$\lim_{t\to\alpha} r(t) = (\lim_{t\to\alpha} x(t))\mathbf{i} + (\lim_{t\to\alpha} y(t))\mathbf{j} + (\lim_{t\to\alpha} z(t))k$$



If the limit of any component does not exist, then we shall agree that the limit of r (t) does not exist. These definitions are also applicable to the one-sided and infinite limits $\lim_{t\to\alpha^+}$, $\lim_{t\to\alpha^-}$ $\lim_{t\to+\infty}$, and $\lim_{t\to-\infty}$. It follows from (1) and (2) that

$$\lim_{t\to\alpha} r(t) = L$$

if and only if the components of r(t) approach the components of L as t → a. Geometrically, this is equivalent to stating that the length and direction of r (t) approach the length and direction of L as t → α

CONTINUITY OF VECTOR-VALUED FUNCTIONS

The definition of continuity for vector-valued functions is similar to that for real-valued functions. We shall say that r is continuous at t_0 if

1. r (t_0) is defined;
2. $\lim_{t\to t_0} r(t)$ exists;
3. $\lim_{t\to t_0} r(t) = r(t_0)$.

It can be shown that r is continuous at t_0 if and only if each component of r is continuous. As with real-valued functions, we shall call r continuous everywhere or simply continuous if r is continuous at al real values of t. geometrically, the graph of a continuous vector-valued function is an unbroken curve.

DERIVATIVES OF VECOR-VALUED FUNCTIONS

The definition of a derivative for vector-valued functions is analogous to the definition for real-valued functions.

DEFINITION

The derivative $r^/(t)$ of a vector-valued function r(t) is defined by

$$\mathbf{r^/(t) = \lim_{h\to 0} \frac{r(t+h) - r(t)}{h}}$$

Provided this limit exists.

For computational purposes the following theorem is extremely useful; it states that the derivative of a vector-valued function can be computed by differentiating each components.

THEOREM

(a) If r(t) = x(t)**i** + y(t)**j** is a vector-valued function in 2-space, and if x(t) and y(t) are differentiable, then

$$r^/(t) = x^/(t)\mathbf{i} + y^/(t)\mathbf{j}$$

(b) If r(t) = x(t)**i** + y(t)**j** + z(t)**k** is a vector-valued function in 3-space, and if x(t), y(t), and z(t) are differentiable, then

$$r'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

We shall prove part (a). The proof of (b) is similar.

Proof (a):

$$\mathbf{r}'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h\to 0}\frac{[x(t+h)-x(t)]}{h}\mathbf{i} + \lim_{h\to 0}\frac{[y(t+h)-y(t)]}{h}\mathbf{j}$$

$$= x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

As with real-valued functions, there are various notations for the derivative of a vector-valued function. If r = r (t), then some possibilities are

$$\frac{d}{dt}[r(t)],\ \frac{dr}{dt},\ \mathbf{r}'(t), \text{ and } \mathbf{r}'$$

EXAMPLE

Let r(t) = t^2i +t^3j. Find r$'$(t) and r$'$(1)

$$\mathbf{r}'(t) = \frac{d}{dt}[t^2]\,\mathbf{i} + \frac{d}{dt}[t^3]\,\mathbf{j}$$

$$= 2t\,\mathbf{i}\ \ 3t^2\,\mathbf{j}$$

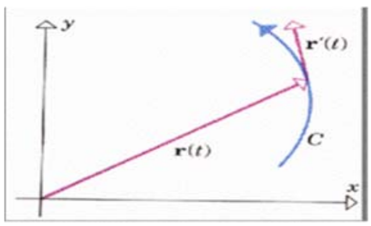
Substituting t=1 yields

$\mathbf{r}'(1) = 2\mathbf{i}+3\mathbf{j}$.

TAGENT VECTORS AND TANGENT LINES

GEOMETRIC INTERPRETATIONS OF THE DERIVATIVE.

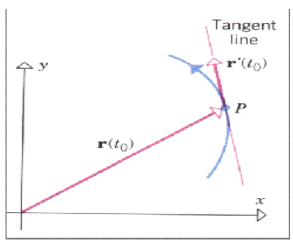
Suppose that C is the graph of a vector-valued function r(t) and that **r**$'$(t) exists and is nonzero for a given value of t. If the vector **r**$'$(t) is positioned with its initial point at the terminal point of the radius vector



DEFINITION

Let P be a point on the graph of a vector-valued function **r**(t), and let **r**(t_0) be the radius vector from the origin to P

If **r**$'$(t_0) exists and **r**$'$(t_0) ≠ **0**, then we call **r**$'$(t_0) the tangent vector to the graph of **r** at **r**(t_0)



REMARKS

Observe that the graph of a vector-valued function can fail to have a tangent vector at a point either because the derivative in (4) does not exist or because the derivative is zero at the point.If a vector-valued function **r**(t) has a tangent vector **r**$'$(t_0) at a point onits graph, then the line that is parallel to **r**$'$(t_0) and passes through the tip of the radius vector **r**(t_0) is called the tangent line of the graph of **r**(t) at **r**(t_0)

Vector equation of the tangent line is

$$\mathbf{r} = \mathbf{r}\,(\mathbf{t_0}) + \mathbf{t}\,\mathbf{r}'(\mathbf{t_0})$$

EXAMPLE

Find parametric equation of the tangent line to the circular helix

$x = \cos t$, $y = \sin t$, $z = 1$ at the point where $t = \pi/6$

To find a vector equation of the tangent line, then we shall equate components to obtain the parametric equations. A vector equation r=r(t) of the helix is

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

Thus, $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

$\Rightarrow \mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + k$

At the point where $t = \pi/6$, these vectors are

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k} \quad \text{and}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}i + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}$$

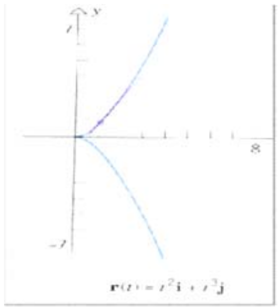
so from (5) with $t_0 = \pi/6$ a vector equation of the tangent line is

$$\mathbf{r}\left(\frac{\pi}{6}\right) + t\,\mathbf{r}'\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k}\right) + t\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}\right)$$

Simplifying, then equating the resulting components with the corresponding components of $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ yields the parametric equation.

$$x = \frac{\sqrt{3}}{2} - \frac{1}{2}t, \quad y = \frac{1}{2} + \frac{\sqrt{3}}{2}t, \quad z = \frac{\pi}{6} + t$$

EXAMPLE



The graph of $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is called a semicubical parabola

Find a vector equation of the tangent line to the graph of $\mathbf{r}(t)$ at

(a) the point (0,0) (b) the point (1,1)

The derivative of $\mathbf{r}(t)$ is

$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

(a) The point (0, 0) on the graph of $\mathbf{r}$ corresponds to $t = 0$. As this point we have $\mathbf{r}'(0) = 0$, so there is no tangent vector at the point and consequently a tangent line does not exist at this point.

(b) The point (1, 1) on the graph of r corresponds to $t = 1$, so from (5) a vector equation of the tangent line at this point is $\mathbf{r} = \mathbf{r}(1) + t\,\mathbf{r}'(1)$

From the formulas for r (t) and $\mathbf{r}'(t)$ with $t = 1$, this equation becomes

$$\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t\,(2\mathbf{i} + 3\mathbf{j})$$

If $\mathbf{r}$ is a vector-valued function in 2-space or 3-space, then we say that $\mathbf{r}(t)$ is smoothly parameterized or that $\mathbf{r}$ is a smooth function of t if the components of r have continuous derivatives with respect to t and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t. Thus, in 3-space $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

is a smooth function of t if $x'(t)$, $y'(t)$, and $z'(t)$ are continuous and there is no value of t at which al three derivatives are zero. A parametric curve C in 2-space or 3-space will be called smooth if it is the graph of some smooth vector-valued function.

It can be shown that a smooth vector-valued function has a tangent line at every point on its graph.

PROPERTIES OF DERIVATIVES
(Rules of Differentiation).
In either 2-space or 3-space let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, and $\mathbf{r}_2(t)$ be vector-valued functions, f(t) a real-valued function, k a scalar, and c a fixed (constant) vector. Then the following rules of differentiation hold:

$$\frac{d}{dt}[c] = 0$$

$$\frac{d}{dt}[kr(t)] = k\frac{d}{dt}[r(t)]$$

$$\frac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)]=\frac{d}{dt}[\mathbf{r}_1(t)]+\frac{d}{dt}[\mathbf{r}_2(t)]$$

$$\frac{d}{dt}[\mathbf{r}_1(t)-\mathbf{r}_2(t)]= \frac{d}{dt}[\mathbf{r}_1(t)] - \frac{d}{dt}[\mathbf{r}_2(t)]$$

$$\frac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\frac{d}{dt}[\mathbf{r}(t)]+\mathbf{r}(t)\frac{d}{dt}[f(t)]$$

In addition to the rules listed in the foregoing theorem, we have the following rules for differentiating dot products in 2-space or 3-space and cross products in 3-space:

$$\frac{d}{dt}[\mathbf{r}_1(t)\mathbf{.}\mathbf{r}_2(t)] = \mathbf{r}_1\mathbf{.}\frac{dr_2}{dt} + \frac{dr_1}{dt}\mathbf{.}\,\mathbf{r}_2 \qquad (6)$$

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\mathbf{r}_1\times\frac{dr_2}{dt}+\frac{dr_1}{dt}\times\mathbf{r}_2 \qquad (7)$$

REMARK:
In (6), the order of the factors in each term on the right does not matter, but in (7) it does.
In plane geometry one learns that a tangent line to a circle is perpendicular to the radius at the point of tangency. Consequently, if a point moves along a circular arc in 2-space, one would expect the radius vector and the tangent vector at any point on the arc to be perpendicular. This is the motivation for the following useful theorem, which is applicable in both 2-space and 3-space.

THEOREM:
If r (t) is a vector-valued function in 2-space or 3-space and ||r(t)|| is constant for all t, then r(t) . r$^{/}$(t) = 0

that is, $\mathbf{r}(t)$ and $\mathbf{r}^{/}(t)$ are orthogonal vectors for all t.It follows from (6) with $\mathbf{r}_1(t)=\mathbf{r}_2(t)=\mathbf{r}(t)$ that

$$\frac{d}{dt}[\mathbf{r}(t).\mathbf{r}(t)] = \mathbf{r}(t).\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\,.\,\mathbf{r}(t)$$

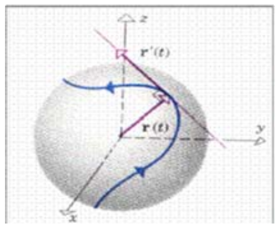
or, equivalently, $\frac{d}{dt}[||\mathbf{r}(t)||]^2 = 2\mathbf{r}(t)\,\mathbf{.}\,\frac{d\mathbf{r}}{dt}$

But $||\mathbf{r}(t)||^2$ is constant, so its derivative is zero. Thus $2r(t).\frac{d\mathbf{r}}{dt} = 0$ that is $\mathbf{r}(t)\,\mathbf{.}\,\frac{d\mathbf{r}}{dt} = 0$

That is the $\mathbf{r}(t)$ is perpendicular $\frac{d\mathbf{r}}{dt}$

EXAMPLE

Just as a tangent line to a circular arc in 2-space is perpendicular to the radius at the point of tangency, so a tangent line to a curve on the surface of a sphere in 3-space is perpendicular to the radius at the point of tangency.



To see that this is so, suppose that the graph of **r**(t) lies on the surface of the sphere of radius k > 0 centered at the origin.For each value of *t* we have ||**r**(t)||=k,

$$\mathbf{r}(t).\ \mathbf{r}^{/}(t) = 0$$

which implies that the radius vector **r**(t) and the tangent vector **r**$^{/}$(t) are perpendicular. This completes the argument because the tangent line, where it exists, is parallel to the tangent vector.

INTEGRALS OF VECTOR VALUED FUNCTION

(a) If r(t)=x(t)I +y(t) j is a vector-valued function in 2-space ,the we define.

$$\int r(t)dt = (\int x(t)dt)i + (\int y(t)dt)\,j \qquad (1a)$$

$$\int_a^b r(t)dt = (\int_a^b x(t)dt)i + (\int_a^b y(t)dt)\,j \qquad (1b)$$

(b) If r(t) = x (t) **i** + y(t) **j** + z(t)k is a vector-valued function in 3-space, then we define.

$$\int \mathbf{r}(t)dt = \left(\int x(t)dt\right)\mathbf{i} + \left(\int y(t)\,dt\right)\mathbf{j} + \left(\int z(t)dt\right)\mathbf{k} \qquad \textbf{(2a)}$$

$$\int_a^b r(t)dt = \left(\int_a^b x(t)dt\right)\mathbf{i} + \left(\int_a^b y(t)dt\right)\mathbf{j} + \left(\int_a^b z(t)dt\right)\mathbf{k} \qquad \textbf{(2b)}$$

$$Let\ r(t) = 2ti + 3t^2 j$$

$$(a)\int r(t)dt \qquad (b)\int_0^2 r(t)dt$$

$$\int r(t)dt = \int (2ti + 3t^2 j)dt = (\int 2tdt)i + (\int 3t^2 dt)\,j$$

$$(t^2 + C_1)i + (t^3 + C_2)\,j = t^2 i + C_1 i + t^3 j + C_2 j = t^2 i + t^3 j + C_1 i + C_2 j = t^2 i + t^3 j + C$$

Where $C = C_1 i + C_2 j$ *is an arbitrary vector constant of integration*

$$(b)\int_0^2 r(t)dt = \int_0^2 (2ti + 3t^2 j)dt = (\int_0^2 2tdt)i + (\int_0^2 3t^2 dt)\,j = \left[t^2\right]_0^2 i + \left[t^3\right]_0^2 j = (2^2 - 0)i + (2^3 - 0)\,j = 4i + 8j$$

PROPERTEIS OF INTEGRALS

$\int cr(t)\,dt = c\int r(t)\,dt$ (3)

$\int [r_1(t)+r_2(t)]\,dt = \int r1(t)dt + \int r_2(t)\,dt$
(4)

$\int [r_1(t)-r_2(t)]dt = \int r_1(t)\,dt - \int r_2(t)\,dt$
(5)

These properties also hold for definite integrals of vector-valued functions. In addition, we leave it for the reader to show that if r is a vector-valued function in 2-space or 3-space, then $\frac{d}{dt}\left[\int r(t)\,dt\right] = r(t)$ (6)

This shows that an indefinite integrals of **r**(t) is, in fact, the set of antiderivatives of r(t), just as for real-valued functions.

If **r**(t) is any antiderivative or **r**(t) in the sense that $\mathbf{R}'(t) = \mathbf{r}(t)$, then

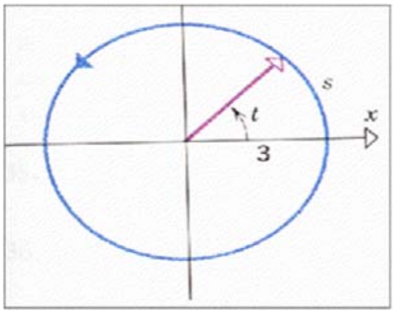
$\int r(t)\,dt = \mathbf{R}(t) + \mathbf{C}$ (7)

where C is an arbitrary vector constant of integration. Moreover,

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

Lecture No -29 Change of parameter

It is possible for different vector-valued functions to have the same graph.
For example, the graph of the function
$r = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$, $0 \le t \le 2\pi$
is the circular of radius 3 centered at the origin with counterclockwise orientation. The parameter t can be interpreted geometrically as the positive angle in radians from the x-axis to the radius vector. For each value of t, let s be the length of the arc subtended by this angle on the circle



The parameters s and t are related by
$t = s/3$, $0 < s < 6\pi$
if we substitute this in (10), we obtain a vector-valued function of the parameter s, namely
$r = 3\cos(s/3)\mathbf{i} + 3\sin(s/3)\mathbf{j}$, $0 \le s \le 6\pi$
whose graph is also the circle of radius 3 centered at the origin with counterclockwise orientation .In various problems it is helpful to change the parameter in a vector-valued function by making an appropriate substitution. For example, we changed the parameter above from t to s by substituting $t=s/3$ in (10).
In general, if g is a real-valued function, then substituting $t = g(u)$ in r(t) changes the parameter from t to u.

When making such a change of parameter, it is important to ensure that the new vector-valued function of u is smooth if the original vector-valued function of t is smooth. It can be proved that this will be so if g satisfies the following conditions:

1. g is differentiable.
2. g' is continuous.
3. $g'(u) \neq 0$ for any u in the domain of g.
4. The range of g is the domain of **r**.

If g satisfies these conditions, then we call $t = g(u)$ a smooth change of parameter. Henceforth, we shall assume that all changes of parameter are smooth, even if it is not stated explicitly.

ARC LENGTH

Because derivatives of vector-valued functions are calculated by differentiating components, it is natural do define integrals of vector-functions in terms of components.

EXAMPLE

If $x'(t)$ and $y'(t)$ are continuous for $a \le t \le b$, then the curve given by the parametric equations
$x = x(t),\ y = y(t)\quad (a \le t \le b)\qquad (9)$
has arc length

$$\mathbf{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (10)$$

This result generalizes to curves in 3-spaces exactly as one would expect:
If $x'(t)$, $y'(t)$, and $z'(t)$ are continuous for $a \le t \le b$, then the curve given by the parametric equations
$x = x(t),\ y = y(t),\ z = z(t)\quad (a \le t \le b)$

has arc length

$$\mathbf{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad (12)$$

EXAMPLE

Find the arc length of that portion of the circular helix

$x = \cos t, \qquad y = \sin t, \qquad z = t$

From $t = 0$ to $t = \pi$

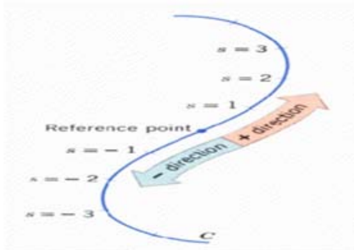
The arc length is

$$\mathbf{L} = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt$$

$$= \int_0^\pi \sqrt{2}\, dt = \sqrt{2}\,\pi$$

ARC LENTH AS A PARAMETER

For many purposes the best parameter to use for representing a curve in 2-space or 3-space parametrically is the length of arc measured along the curve from some fixed reference point. This can be done as follows:



Step 1: Select an arbitrary point on the curve C to serve as a reference point.

Step 2: Starting from the reference point, choose one direction along the curve to be the positive direction and the other to be the negative direction.

Step 3: If P is a point on the curve, let s be the "signed" arc length along C from the reference point to P, where s is positive if P is in the positive direction from the reference point, and s is negative if P is in the negative direction.

By this procedure, a unique point P on the curve is determined when a value for s is given. For example, $s = 2$ determines the point that is 2 units along the curve in the positive direction from the reference point, and $s = -\frac{3}{2}$ determines the point that is $\frac{3}{2}$ units along the curve in the negative direction from the reference point.

Let us now treat s as a variable. As the value of s changes, the corresponding point P moves along C and the coordinates of P become functions of s. Thus, in 2-space the coordinates of P are (x(x,), y(s)), and in 3-space they are (x(s), y(s), z(s)). Therefore, in 2-space the curve C is given by the parametric equations $x = x(s), \quad y = y(s)$

and in 3-space by $x = x(s), \quad y = y(s), \quad z = z(s)$

REMARKS

When defining the parameter s, the choice of positive and negative directions is arbitrary. However, it may be that the curve C is already specified in terms of some other parameter t, in which case we shall agree always to take the direction of increasing t as the positive direction for the parameter s. By so doing, s will increase as t increases and vice versa.

The following theorem gives a formula for computing an arc-length parameter s when the curve C is expressed in terms of some other parameter t. This result will be used when we want to change the parameterization for C from t to s.

THEOREM

(a) Let C be a curve in 2-space given parametrically by

$x = x(t)$, $y = y(t)$

where $x'(t)$ and $y'(t)$ are continuous functions. If an arc-length parameter s is introduced with its reference point at $(x(t_0), y(t_0))$, then the parameters s and t are related by

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \qquad (13a)$$

(b) Let C be a curve in 3-space given parametrically by

$x = x(t)$, $y = y(t)$, $z = z(t)$

where $x'(t)$, $y'(t)$, and $z'(t)$ are continuous functions. If an arc-length parameter s is introduced with its reference point at $(x(t_0), y(t_0), z(t_0))$, then the parameters s and t are related by

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \qquad (13b)$$

Proof

If $t > t_0$, then from (10) (with *u* as the variable of integration rather than *t*) it follows that

$$\int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \qquad (14)$$

represents the arc length of that portion of the curve C that lies between $(x(t_0), y(t_0))$ and $(x(t), y(t))$. If $t < t_0$, then (14) is the negative of this arc length. In either case, integral (14) represents the "signed" arc length s between these points, which proves (13a).

It follows from Formulas (13a) and (13b) and the Second Fundamental Theorem of Calculus (Theorem 5.9.3) that in 2-space.

$$\frac{ds}{dt} = \frac{d}{dt}\left[\int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du\right] = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

and in 3-space

$$\frac{ds}{dt} = \frac{d}{dt}\left[\int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, dt\right] = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Thus, in 2-space and 3-space, respectively,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad (15a)$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \qquad (15b)$$

REMARKS:

Formulas (15a) and (15b) reveal two facts worth noting. First, ds/dt does not depend on t_0; that is, the value of ds/dt is independent of where the reference point for the parameter s is located. This is to be expected since changing the position of the reference point shifts each value of s by a constant (the arc length between the reference points), and this constant drops out when we differentiate. The second fact to be noted from (15a) and (15b) is that $ds/dt \geq 0$ for all t. This is also to be expected since s increases with t by the remark preceding Theorem 15.3.2. If the curve C is smooth, then it follows from (15a) and (15b) that $ds/dt \geq 0$ for all t .

EXAMPLE

$x = 2t + 1,\ \ y = 3t - 2$ (16)

using arc length s as a parameter, where the reference point for s is the point (1, – 2).

In formula (13a) we used u as the variable of integration because t was needed as a limit of integration. To apply (13a), we first rewrite the given parametric equations with u in place of t; this gives

from which we obtain

$x = 2u + 1, \qquad y = 3u - 2$

$\dfrac{dx}{du} = 2, \qquad \dfrac{dy}{du} = 3$

we see that the reference point (1,–2) corresponds to $t = t_0 = 0$

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \int_{t_0}^{t} \sqrt{13}\ du = \sqrt{13}u \Big]_{u=0}^{u=t} = \sqrt{13}t$$

Therefore, $t = \dfrac{1}{\sqrt{13}}\, s$

Substituting this expression in the given parametric equations yields.

$$x = 2\left(\frac{1}{\sqrt{13}}s\right) + 1 = \frac{2}{\sqrt{13}}s + 1$$

$$y = 3\left(\frac{1}{\sqrt{13}}s\right) - 2 = \frac{3}{\sqrt{13}}s - 2$$

EXAMPLE

Find parametric equations for the circle $x = a\cos t,\ y = a\sin t\ \ (0 \le t \le 2\pi)$

using arc length s as a parameter, with the reference point for s being (a, 0), where a > 0.

We first replace t by u in the given equations so that $x = a\cos u,\ \ y = a\sin u$

And $\dfrac{dx}{du} = -a\sin u, \ \ \dfrac{dy}{du} = a\cos u$

Since the reference point (a, 0) corresponds to t = 0, we obtain

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \int_{t_0}^{t} \sqrt{(-a\sin u)^2 + (a\cos u)^2}\ du = \int_{0}^{t} a\, du = au \Big]_{u=0}^{u=t} = at$$

Solving for t in terms of s yields $t = s/a$

Substituting this in the given parametric equations and using the fact that $s = at$ ranges from 0 to $2\pi a$ as t ranges from 0 to 2π, we obtain

$x = a\cos(s/a),\ \ y = a\sin(s/a)\ (0 \le s \le 2\pi a)$

Example

Find Arc length of the curve $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + \tfrac{1}{2}\sqrt{6}\, t^2\, \mathbf{k},\ 1 \le t \le 3$

Here $x = t^3,\ y = t,\ z = \tfrac{1}{2}\sqrt{6}\, t^2$

$\dfrac{dx}{dt} = 3t^2,\ \dfrac{dy}{dt} = 1,\ \dfrac{dz}{dt} = \sqrt{6}\, t$

$$\text{Arc length} = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_1^3 \sqrt{9t^4 + 1 + 6t^2}\ dt = \int_1^3 \sqrt{(3t^2+1)^2}\ dt$$

$$= \left| t^3 + t \right|_1^3 = (3)^3 + 3 - (1)^3 - 1 = 27 + 3 - 1 - 1 = 28$$

<u>EXAMPLE</u>

Calculate $\frac{d\mathbf{r}}{du}$ by chain Rule.

$\mathbf{r} = e^t\mathbf{i} + 4e^{-t}\mathbf{j}$

$\frac{dr}{dt} = e^t\mathbf{i} - 4e^{-t}\mathbf{j}$

$\frac{dt}{du} = 2u$

$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{du} = (e^t\mathbf{i} - 4e^t\mathbf{j}).(2u) = 2u\, e^{u^2}\mathbf{i} - 8ue^{-u^2}\mathbf{j}$

By expressing $\mathbf{r}$ in terms of u

$R = e^u\mathbf{i} + 4e^{-u^2}\mathbf{j}$

$\frac{d\mathbf{r}}{du} = 2u\, e^{u^2}\mathbf{i} - 8ue^{-u^2}\mathbf{j}$

Lecture No -30 Exact Differential

If z = f (x, y), then $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

The result can be extended to functions of more than two independent variables.

If z = f(x, y, w), $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial w} dw$

Make a note of these results in differential form as shown.

Exercise

Determine the differential dz for each of the following functions.

1. $z = x^2 + y^2$
2. $z = x^3 \sin 2y$
3. $z = (2x - 1) e^{3y}$
4. $z = x^2 + 2y^2 + 3w^2$
5. $z = x^3y^2 w$.

Finish all five and then check the result.

---

1. $dz = 2 (x\, dx + y\, dy)$
2. $dz = x^2 (3 \sin 2y\, dx + 2x \cos 2y\, dy)$
3. $dz = e^{3y} \{2dx + (6x - 3) dy\}$
4. $dz = 2 (xdx + 2ydy + 3wdw)$
5. $dz = x^2y (3ywdx + 2xwdy + xydw)$

Exact Differential

We have just established that if z=f(x, y)

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

We now work in reverse.

Any expression dz = Pdx + Qdy, where P and Q are functions of x and y, is an exact differential if it can be integrated to determine z.

$\therefore\; P = \frac{\partial z}{\partial x}$ and $Q = \frac{\partial z}{\partial y}$

Now $\frac{\partial P}{\partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y}$ and we know that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

Therefore, for dz to be an exact differential $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and this is the test we apply.

Example

$dz = (3x^2 + 4y^2)\, dx + 8xy\, dy$.

If we compare the right-hand side with Pdx + Qdy, then

$P = 3x^2+4y^2 \;\; \therefore \;\; \frac{\partial P}{\partial y} = 8y$

$Q = 8xy \qquad \therefore \;\; \frac{\partial Q}{\partial x} = 8y$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \therefore$ dz is an exact differential

Similarly, we can test this one.

Example

$dz = (1 + 8xy)\, dx + 5x^2\, dy.$

From this we find dz is not an exact differential

for $dz = (1 + 8xy)\, dx + 5x^2\, dy$

$\therefore\ P = 1 + 8xy \quad \therefore\ \dfrac{\partial P}{\partial y} = 8x$

$Q = 5x^2 \quad \therefore\ \dfrac{\partial Q}{\partial x} = 10x$

$\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ $\therefore$ dz is not an exact differential

Exercise

Determine whether each of the following is an exact differential.

1. $dz = 4x^3y^3dx + 3x^4y^2\, dy$
2. $dz = (4x^3y+2xy^3)\, dx+(x^4+3x_2y^2)\, dy$
3. $dz=(15y^2e^{3x}+2xy^2)dx+(10ye^{3x}+x^2y)dy$
4. $dz=(3x^2e^{2y}-2y^2e^{3x})dx+(2x^3e^{2y}-2ye^{3x})dy$
5. $dz=(4y^3\cos 4x+3x^2\cos 2y)dx+(3y^2\sin 4x-2x^3\sin 2y)\, dy.$

1. Yes 2. Yes 3. No 4. No 5. Yes

We have just tested whether certain expressions are, in fact, exact differentials−and we said previously that, by definition, an exact differential can be integrated. But how exactly do we go about it? The following examples will show.

Integration Of Exact Differentials

$dz = Pdx+Qdy$ where $P=\dfrac{\partial z}{\partial x}$ and $Q=\dfrac{\partial z}{\partial y}$

$\therefore\ z = \int Pdx$ and also $z = \int Qdy$

Example

$dz = (2xy + 6x)\, dx + (x2 + 2y3)\, dy.$

$P = \dfrac{\partial z}{\partial x} = 2xy + 6x \quad \therefore\ z=\int (2xy+6x)dx$

$\therefore\ z = x^2y + 3x^2 + f(y)$ where f(y) is an arbitrary function of y only, and is akin to the constant of integration in a normal integral.

Also

$Q = \dfrac{\partial z}{\partial y} = x^2 + 2y^3 \quad \therefore\ z = \int (x^2+2y^3)\, dy$

$\therefore\ z = x2y + \dfrac{y^4}{2} + F(x)$ where F(x) is an arbitrary function of x only.

$z = x^2y + 3x^2 + f(y) \quad$ (i)

and $\quad z = x^2y + \dfrac{y^4}{2} +F(x) \quad$ (ii)

For these two expressions to represent the same function, then

f(y) in (i) must be $\dfrac{y^4}{2}$ already in (i)

and F(x) in (ii) must be $3x^2$ already in (i)

$\therefore\ z = x^2y + 3x^2 + \dfrac{y^4}{2}$

Example

Integrate $dz = (8e^{4x} + 2xy^2)\,dx + (4\cos 4y + 2x^2y)\,dy$.

Argue through the working in just the same way, from which we obtain

$z = 2e^{4x} + x^2y^2 + \sin 4y$

Here it is.

$dz = (8e^{4x} + 2xy^2)\,dx+(4\cos4y+2x^2y)\,dy$

$P = \dfrac{\partial x}{\partial x} = 8e^{4x} + 2xy^2$

$\therefore\ z = \int (8e^{4x} + 2xy^2)\,dx$

$\therefore\ z = 2e^{4x} + x^2y^2 + f(y)$ (i)

$Q = \dfrac{\partial z}{\partial y} = 4\cos 4y + 2x^2y$

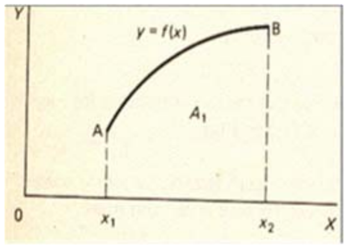
$\therefore\ z = \int (4\cos 4y + 2x^2y)\,dy$

$\therefore\ z = \sin 4y + x^2y^2 + F(x)$ (ii)

For (i) and (ii) to agree, $f(y) = \sin 4y$ and $F(x) = 2e^{4x}$

$\therefore\ z = 2e^{4x} + x^2y^2 + \sin 4y$

Area enclosed by the closed curve

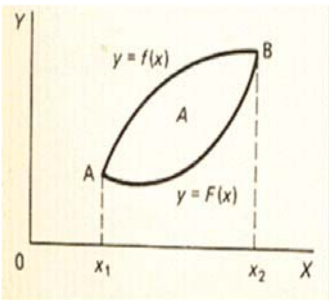


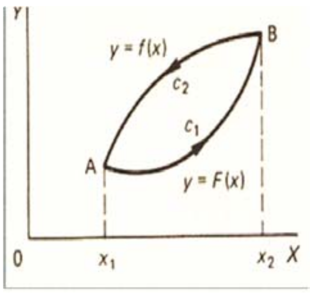
One of the earliest application of integration is finding the area of a plane figure bounded by the x-axis, the curve $y = f(x)$ and ordinates at $x=x_1$ and $x=x_2$.

$A_1=\int_{x_1}^{x_2} y\,dx=\int_{x_1}^{x_2} f(x)\,dx$

If points A and B are joined by another curve, $y = F(x)$

$A_2 = \int_{x_1}^{x_2} f(x)\,dx$





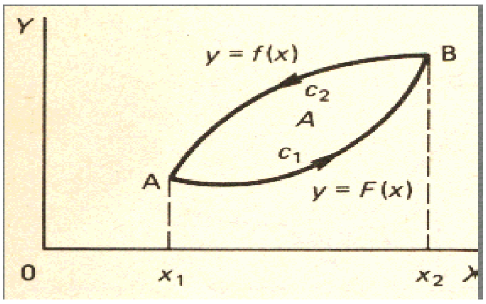
Combining the two figures, we have

$A=A_1-A_2 \qquad \therefore\ A=\int_{x_1}^{x_2} f(x)\,dx-\int_{x_1}^{x_2} f(x)\,dx$

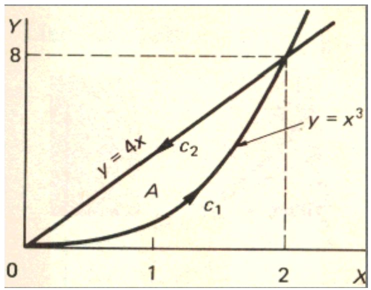
The final result above can be written in the form

$$A = -\oint y\,dx$$

Where the symbol $\oint$ indicates that the integral is to be evaluated round the closed boundary in the positive



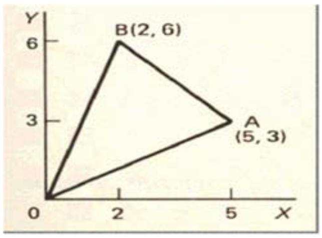
Example

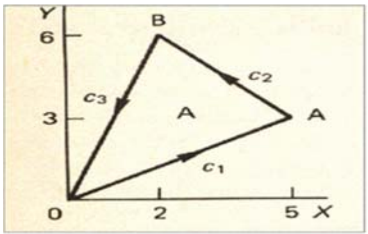


Determine the area enclosed by the graph of $y = x^3$ and $y = 4x$ for $x \geq 0$.
First we need to know the points of intersection. These are $x = 0$ and $x = 2$
We integrate in a an anticlockwise manner
c_1: $y = x^3$, limits $x = 0$ to $x = 2$
c_2: $y = 4x$, limits $x = 2$ to $x = 0$.
$A = -\oint y\,dx = A = 4$ square units

For $A = -\oint y dx = -\left\{\int_1^2 x^3 dx + \int_2^0 4x dx\right\} = -\left\{\left(\frac{x^4}{4}\right)_0^2 + \left[2x^2\right]_0^2\right\} = 4$

Example
Find the area of the triangle with vertices (0, 0), (5, 3) and (2, 6).





The equation of OA is $y = \frac{3}{5}x$, BA is $y = 8 - x$, OB is $y = 3x$

Then $A = -\oint y\,dx$
Write down the component integrals with appropriate limits.

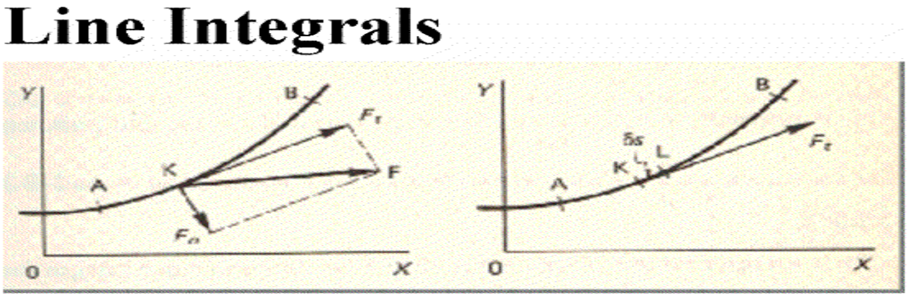
$$A = -\oint y dx = -\left\{\int_0^5 \frac{3}{5}x dx + \int_5^2 (8-x)dx + \int_2^0 3x dx\right\}$$

The limits chosen must progress the integration round the boundary of the figure in an anticlockwise manner. Finishing off the integration, we have
A = 12 square units

The actual integration is easy enough. The work we have just done leads us on to consider line integrals, so let us make a fresh start in the next frame.

<u>Line Integrals</u>

If a field exists in the xy-plane, producing a force F on a particle at K, then F can be resolved into two components.F_1 along the tangent to the curve AB at K. Fa along the normal to the curve AB at K.



The work done in moving the particle through a small distance δs from K to L along the curve is then approximately F_1 δs. So the total work done in moving a particle along the curve from A to B is given by

$$\lim_{\delta \to 0} \sum F_t \,\delta s = \int F_t \, ds \text{ from A to B}$$

This is normally written $\int_{AB} F_t \, ds$ where A and B are the end points of the curve,

or as $\int_C F_t \, ds$ where the curve c connecting A and B is defined.

Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve c joining A and B.

$$\therefore \quad I = \int_{AB} F_t \, dx = \int_C F_t \, ds$$

where c is the curve y = f(x) between A(x_1, y_2) and B (x_2, y_2).

There is in fact an alternative form of the integral which is often useful, so let us also consider that.

<u>Alternative form of a line integral</u>

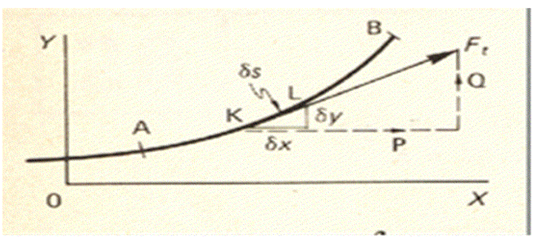
It is often more convenient to integrate with respect to x or y than to take arc length as the variable.

If F_t has a component

P in the x-direction

Q in the y-direction

then the work done from K to L can be stated as Pδx + Qδy

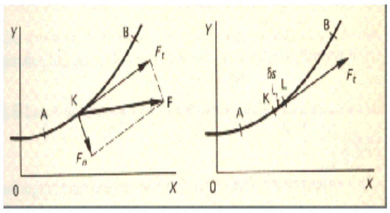


$\therefore \int_{AB} F_t \, ds = \int_{AB} (P\,dx + Q dy)$
where P and Q are functions of x and y.
In general then, the line integral can be expressed as

$$I = \int_C F_t \, ds = \int_C (P\,dx + Q dy)$$

where c is the prescribed curve and F, or P and Q, are functions of x and y.
Make a note of these results –then we will apply them to one or two examples.

Lecture No -31 Line Integral



The work done in moving the particle through a small distance δs from K to L along the curve is then approximately $F_1\ \delta s$. So the total work done in moving a particle along the curve from A to B is given by

$$\lim_{\delta \to 0} \sum F_t\, \delta s = \int F_t\, ds \text{ from A to B}$$

This is normally written $\int_{AB} F_t\, ds$ where A and B are the end points of the curve, or as $\int_C F_t\, ds$ where the curve c connecting A and B is defined.Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve c joining A and B.

$$\therefore\ I = \int_{AB} F_t\, dx = \int_C F_t\, ds$$

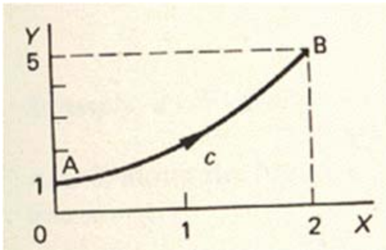
where c is the curve y = f(x) between A(x_1, y_2) and B (x_2, y_2).
There is in fact an alternative form of the integral which is often useful, so let us also consider that.

Alternative form of a line integral

It is often more convenient to integrate with respect to x or y than to take arc length as the variable.
If F_t has a component ,P in the x-direction ,Q in the y-direction
then the work done from K to L can be stated as $P\delta x + Q\delta y$

Example 1

Evaluate $\int_C (x + 3y)\, dx$ from A (0, 1) to B (2, 5)
along the curve $y = 1 + x^2$.

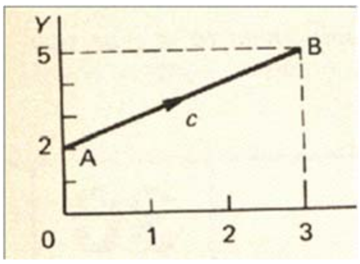


The line integral is of the form
$\int_C (P\, dx + Q dy)$ where, in this case, Q = 0 and c is the curve $y = 1 + x^2$.

It can be converted at once into an ordinary integral by substituting for y and applying the appropriate limits of x.

$$I = \int_C (Pdx+Qdy) = \int_C (x+3y)dx = \int_C^{2} (x+3+3x^2)dx$$

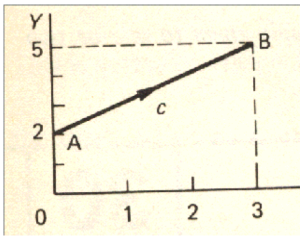
$$= \left[\frac{x^2}{2} + 3x + x^3\right]_0^2 = 16$$



Now for another, so turn on.

Example 2

Evaluate $I = \int_C (x^2 + y)\,dx + (x - y^2)dy$ from A (0, 2) to B (2, 5) along the curve y = 2 + x.



$I = \int_C (Pdx + Qdy)$

$P = x^2 + \ y = x^2 + 2 + x = x^2 + x + 2$

$Q = x - y^2 = x - (4+4x+x^2) = -\,(x^2+3x+4)$

Also y = 2 + x

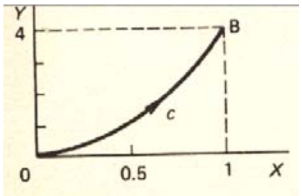
$\therefore$ dy = dx and the limits are x=0 to x=3

$\therefore$ I = – 15

for $I = \int_0^2 \{(x^2+x+2)\,dx - (x^2+3x+4)\,dx\}$ or $\int_0^2 -(2x+2)\,dx = \left[-x^2 - 2x\right]_0^2 = -4 - 4 = -8$

Example 3

Evaluate $I = \int_C \{(x^2+2y)dx + xydy\}$ from O(0, 0) to B(1, 4) along the curve $y=4x^2$.



In this case, c is the curve y = 4x2.

$\therefore$ dy = 8x dx

Substitute for y in the integral and apply the limits.

Then I = 9.4

for $\quad I = \int_C \{(x^2+2y)\,dx+xydy\}$

$y = 4x^2 \qquad \therefore\ dy = 8xdx$

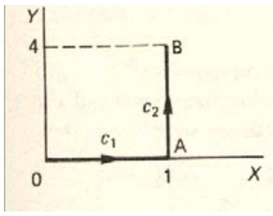
also $x^2 + 2y = x^2 + 8x^2 = 9x^2$; $\ xy = 4x^3$

$\therefore\ I=\int_0^1 \{9x^2dx+32x^4dx\}=\int_0^1 (9x^2+32x^4)\,dx = 9.4$

They are all done in very much the same way.

Example 4

Evaluate $I = \int_C \{(x^2 + 2y)\,dx + xydy\}$ from O(0, 0) to A (1, 0) along the line y = 0 and then from A (1, 0) to B (1, 4) along the line x = 1.



(i) OA : c1 is the line y = 0 $\quad\therefore$ dy = 0.

Substituting y = 0 and dy = 0 in the given integral gives.

$$I_{OA} = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(ii) AB: Here c_2 is the line x = 1 $\therefore$ dx=0

$\therefore\ I_{AB} = 8$

$$\text{For } I_{AB} = \int_0^4 \{(1 + 2y)(0) + ydy\} = \int_0^4 ydy = \left[\frac{y^2}{2}\right]_0^4 = 8$$

$$\text{Then } I = I_{OA}+I_{AB} = \frac{1}{3} + 8 = 8\frac{1}{3} \qquad \therefore\ I=8\frac{1}{3}$$

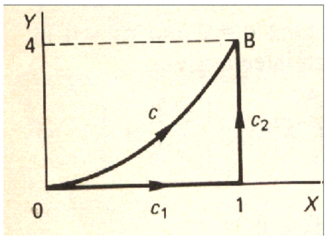
If we now look back to Example 3 and 4 just completed, we find that we have evaluated the same integral between the same two end points, but along different paths of integration.If we combine the two diagrams, we have

where c is the curve $y = 4x^2$ and $c1 + c^2$ are the lines

y = 0 and x = 1. The result obtained were

$I_c = 9\frac{2}{3}$ and $I_{c_1+c_2} = 8\frac{1}{3}$

Notice therefore that integration along two distinct paths joining the same two end points does not necessarily give the same results.



Properties of line integrals

1. $\int_C F ds = \int_C \{P dx + Q dy\}$

2. $\int_{AB} F ds = -\int_{BA} F ds$ and $\int_{AB} \{Pdx+Qdy\} = \int_{BA} \{Pdx+Qdy\}$

i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.

3. (a) For a path of integration parallel to the y-axis, i.e. $x = k$, $dx = 0$

$\therefore \int_C P dx = 0 \quad \therefore I_C = \int_C Q dy.$

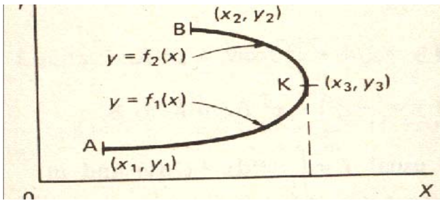
(b) For a path of integration parallel to the x-axis, i.e. $y = k$, $dy = 0$.

$\therefore \int_C Q dy = 0 \therefore I_C = \int_C P dx.$

4. If the path of integration c joining A to B is divided into two parts AK and KB, then $I_c = I_{AB} = I_{AK} + I_{KB}$.

5 .If the path of integration c is not single valued for part of its extent, the path is divided into two sections.

$y = f_1(x)$ from A to K, $y = f_2(x)$ from K to B.



6. In all cases, the actual path of integration involved must be continuous and single-valued.

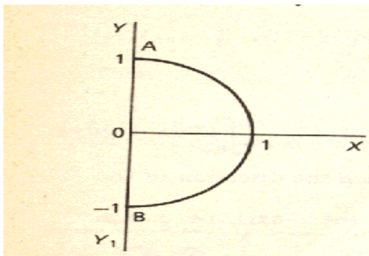
Example

Evaluate $I = \int_C (x + y) dx$ from A(0, 1) to B (0, − 1) along the semi-circle $x^2+y^2=1$ for $x \geq 0$. The first thing we notice is that the path of integration c is not single-valued

For any value of x, $y = \pm\sqrt{1 - x^2}$. Therefore, we divided c into two parts

(i) $y = \sqrt{1 - x^2}$ from A to K

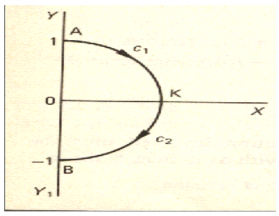
(ii) $y = -\sqrt{1 - x^2}$ from K to B



As usual, $I = \int_C (Pdx + Qdy)$ and in this particular case, $Q = 0$

$$\therefore I = \int_C P dx = \int_0^1 (x+\sqrt{1-x^2})\, dx + \int_1^0 (x\sqrt{1-x^2})\, dx$$

$$= \int_0^1 (x + \sqrt{1-x^2} - x + \sqrt{1-x^2}\, dx = 2\int_0^1 \sqrt{1-x^2}\, dx$$



Now substitute $x = \sin\theta$ and finish it off. $I = \frac{\pi}{2}$

for $I = 2\int_0^1 \sqrt{1-x^2}dx$ $\quad x = \sin\theta$

$\therefore \quad dx = \cos\theta\, d\theta \quad \sqrt{1-x^2} = \cos\theta$

Limits : $x = 0, \quad \theta = 0; \; x = 1, \; \theta = \frac{\pi}{2}$

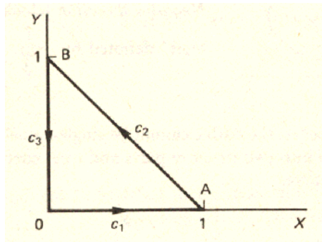
$\therefore \quad I = 2\int_0^{\pi/2} \cos2\theta d\theta = \int_0^{\pi/2} (1+\cos2\theta)d\theta = \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$

Now let us extend this line of development a stage further.

Example

Evaluate the line integral

$I = \oint (x^2dx - 2xy\, dy)$ where c comparises the three sides of the triangle joining O(0, 0), A (1, 0) and B (0, 1).



First draw the diagram and mark in c_1, c_2 and c_3, the proposed directions of integration. Do just that.The three sections of the path of integration must be arranged in an anticlockwise manner round the figure.

Now we deal with each pat separately.

(a) OA : c_1 is the line $y = 0$

Therefore, $dy = 0$.

Then $I = \oint (x^2dx - 2xy\, dy)$ for this part becomes

$I_1 = \int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$ therefore $I_1 = \frac{1}{3}$

(b) AB : for c_2 is the line $y = 1 - x$

$\therefore$ $dy = - dx$.

$I_2 = \int_0^1 \{x^2dx+2x(1-x)dx\} = \int_0^1 (x^2+2x-2x^2)dx = \int_0^1 (2x-x^2)\, dx = \left[x^2 - \frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$

$\therefore \quad I_2 = -\frac{2}{3}$

Note that anticlockwise progression is obtained by arranging the limits in the appropriate order.

Now we have to determine I_3 for BO.

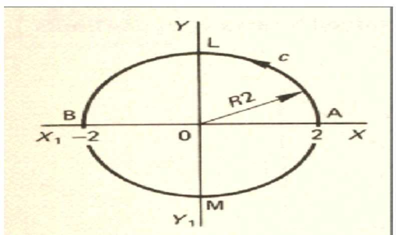
(c) BO: c_3 is the line $x = 0$

$\therefore \; dx = 0 \qquad \therefore I_3 = \int 0dy = 0 \;\; \therefore \;\; I_3 = 0$

Finally, $I = I_1+I_2+I_3 = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3} \qquad \therefore \quad I = -\frac{1}{3}$

Example

Evaluate $\oint_c y\, dx$ when c is the circle $x^2+y^2 = 4$.



$x^2 + y^2 = 4 \qquad \therefore \quad y = \pm\sqrt{4 - x^2}$

y is thus not single-valued. Therefore use

$y = \sqrt{4 - x^2}$ for ALB between

$x = 2$ and $x = -2$ and

$y = -\sqrt{4 - x^2}$ for BMA between

$x = -2$ and $x = 2$.

$$\therefore \quad I = \int_{2}^{-2} \sqrt{4 - x^2}\, dx + \int_{-2}^{2} \{-\sqrt{4 - x^2}\}\, dx$$

$$= 2\int_{2}^{-2} \sqrt{4 - x^2}\ dx = -2\int_{-2}^{2} \sqrt{4 - x^2}\, dx \qquad = -4\int_{0}^{2} \sqrt{4 - x^2}\, dx.$$

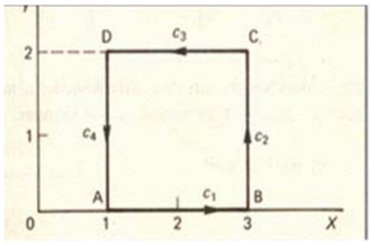
To evaluate this integral, substitute $x = 2 \sin \theta$ and finish it off. $I = -4\pi$

Lecture No -32 Examples

Example

Evaluate $I = \oint$ {xydx+(1+y²)dy} where c is the boundary of the rectangle joining A(1,0), B (3, 0), C(3, 2), D (1, 2).

First draw the diagram and insert c_1, c_2, c_3, c_4.



That give

Now evaluate I_1 for AB; I_2 for BC; I_3 for CD; I_4 for DA; and finally I.

$I_1 = 0$; $I_2 = 4\frac{2}{3}$; $I_3 = -8$ $I_4 = -4\frac{2}{3}$; $I = -8$

Here is the complete working.

$I = \oint \{xydx + (1 + y^2)\, dy\}$

(a) AB: c_1 is $y = 0$ $\therefore$ $dy = 0$ $\therefore$ $I_1 = 0$

(b) BC: c_2 is $x = 3$ $\therefore$ $dx = 0$

$$\therefore I_2 = \int_0^2 (1+y^2)dy = \left[y + \frac{y^3}{3}\right]_0^2 = 4\frac{2}{3} \quad \therefore I_2 = 4\frac{2}{3}$$

(c) CD : c_3 is $y = 2$ $\therefore$ $dy = 0$

$$\therefore I_3 = \int_3^1 2xdx = \left[x^2\right]_3^1 = -8 \quad \therefore I_3 = -8$$

(d) DA : c_4 is $x = 1$ $\therefore$ $dx = 0$

$$\therefore I_4 = \int_2^0 (1+y^2)\, dy = \left[y + \frac{y^3}{3}\right]_2^0 = -4\frac{2}{3}$$

Finally $I = I_1 + I_2 + I_3 + I_4 = 0 + 4\frac{2}{3} - 8 - 4\frac{2}{3} = -8 \; \therefore I = -8$

Remember that, unless we are directed otherwise, we always proceed round the closed boundary in an anticlockwise manner.

Line integral with respect to arc length

We have already established that

$$I = \int_{AB} F_t ds = \int_{AB} \{Pdx+Qdy\}$$

where F_t denoted the tangential force along the curve c at the sample point K(x,y).

The same kind of integral can, of course, relate to any function f(x,y) which is a function of the position of a point on the stated curve, so that

$$I = \int_C f(x, y)\, ds.$$

This can readily be converted into an integral in terms of x:

$$I = \int_C f(x,y)dx = \int_C f(x,y)\frac{ds}{dx}\, dx$$

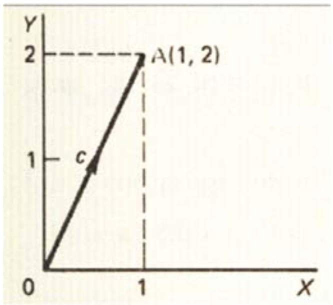
where $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$\therefore \int_C f(x,y)\, dx = \int_{x_1}^{x_2} f(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \text{------------------ } (1)$$

Example

Evalaute $I = \int_C (4x+3xy)ds$ where c is the straight line joining O(0,0) to A (1,2).

c is the line y = 2x $\therefore \frac{dy}{dx} = 2$



$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{5}$$

$$\therefore I = \int_{x=0}^{x=1} (4x+3xy)ds = \int_0^1 (4x+3xy)(\sqrt{5})\, dx. \text{ But } y = 2x$$

$$\text{for} \quad I = \int_0^1 (4x+6x^2)(\sqrt{5})\, dx = 2\sqrt{5}\int_0^1 (2x+3x^2)\, dx = 4\sqrt{5}$$

Parametric Equations

When x and y are expressed in parametric form, e.g. x = y (t), y = g(t), then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \therefore\ ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$I = \int_C f(x,y)ds = \int_{t_1}^{t_2} f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \text{ ----------------(2)}$$

Example

Evaluate $I = \oint 4xyds$ where c is defined as the curve x = sin t, y = cos t between t=0 and $t=\frac{\pi}{4}$.

We have x = sin t $\quad \therefore \frac{dx}{dt} = \cos t$, y = cos t $\quad \therefore \frac{dy}{dt} = -\sin t$

$$\therefore \quad \frac{ds}{dt} = I$$

$$\text{for } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\therefore I = \int_{t_1}^{t_2} f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/4} 4 \sin t \cos t\, dt = 2\int_0^{\pi/4} \sin 2t\, dt$$

$$= -2\left[\frac{\cos 2t}{2}\right]_0^{\pi/4} = 1$$

Dependence of the line integral on the path of integration

We know that integration along two separate paths joining the same two end points does not necessarily give identical results.With this in mind, let us investigate the following problem.

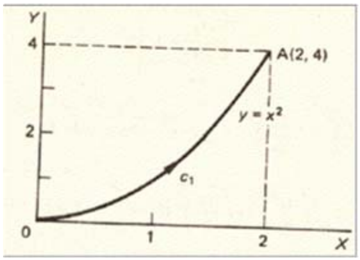
EXAMPLE

Evaluate $I = \oint_C \{3x^2 y^2\, dx + 2x^3 y\, dy\}$ between O (0, 0) and A (2, 4)

(a) along c_1 i.e. $y = x^2$
(b) along c_2 i.e. $y = 2x$
(c) along c_3 i.e. $x = 0$ from (0,0) to (0,4) and $y = 4$ from (0,4) to (2,4).

(a).First we draw the figure and insert relevant information.

$$I = \int_C \{3x^2y^2dx + 2x^3ydy\}$$



The path c_1 is $y = x^2 \quad \therefore\ dy = 2x\, dx$

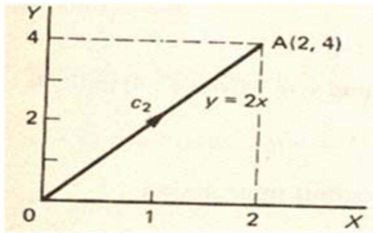
$$\therefore\ I_1 = \int_0^2 \{3x^2x^4dx+2x^3x^22xdx\}=\int_0^2 (3x^6 + 4x^6)\, dx$$

$$\therefore\ = \left[x^7\right]_0^2 = 128 \quad \therefore\ I_1 = 128$$

(b) In (b), the path of integration changes to c_2, i.e. $y = 2x$
So, in this case,for with c_2, $y = 2x \quad \therefore\ dy = 2dx$

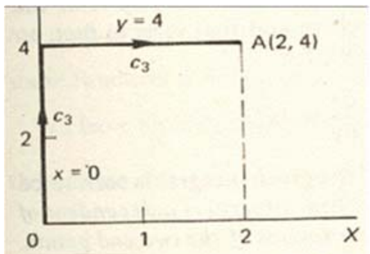
$$\therefore\ I_2 = \int_0^2 (3x^2\, 4x^2\, dx + 2x^3\, 2x^2\, dx\}$$

$$= \int_0^2 20\, x^4\, dx = 4\left[x^5\right]_0^2 = 128 \qquad \therefore\ I_2 = 128$$

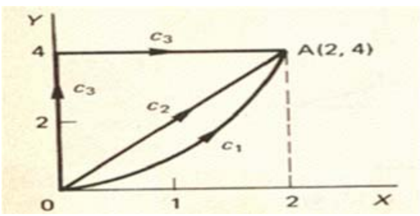


(c) In the third case, the path c_3 is split
$x = 0$ from (0,0) to (0, 4), $y = 4$ from (0, 4) to (2, 4)
Sketch the diagram and determine I_3.
from (0,0) to (0,4) $x=0 \ \therefore\ dx=0 \ \therefore\ I_{3a}=0$
from (0,4) to (2,4) $y=4 \ \therefore\ dy=0 \ \therefore\ I_{3b}=48$

$$\int_0^2 48x^2\, dx = 128 \qquad \therefore\ I_3 = 128$$



In the example we have just worked through, we took three different paths and in each case, the line integral produced the same result. It appears, therefore, that in this case, the value of the integral is independent of the path of integration taken.



We have been dealing with $I = \int_C \{3x^2y^2dx+2x^3ydy\}$

On reflection, we see that the integrand $3x^2 y^2\, dx + 2x^3y\, dy$ is of the form Pdx+Qdy which we have met before and that it is, in fact, an exact differential of the function

$z = x^3y^2$, for $\frac{\partial z}{\partial x} = 3x^2 y^2$ and $\frac{\partial z}{\partial y} = 2x^3 y$

This always happens. If the integrand of the given integral is seen to be an exact differential, then the value o the line integral is independent of the path taken and depends only on the coordinates of the two end points.

Lecture No -33 Examples

Example

Evaluate $I = \int_C \{3ydx + (3x+2y)dy\}$ from A(1, 2) to B (3, 5).

No path is given, so the integrand is doubtless an exact differential of some function z = f (x,y). In fact $\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$. We have already dealt with the integration of exact differentials, so there is no difficulty. Compare with

$I = \int_C \{P\,dx + Q\,dy\}$.

$P = \frac{\partial z}{\partial x} = 3y \qquad \therefore\ z = \int 3ydz = 3xy + f(y)$ -------- (i)

$Q = \frac{\partial z}{\partial y} = 3x + 2y \qquad \therefore\ z = \int (3x+2y)\,dy = 3xy + y^2 + F(x)$ (ii)

For (i) and (ii) to agree $f(y) = y^2$; $F(x) = 0$

Hence z = 3xy + y2

$$\therefore I = \int_C \{3ydx + (3x+2y)dy\} = \int_{(1,2)}^{(3,5)} d(3xy+y^2) = \left[3xy+y^2\right]_{(1,2)}^{(3,5)} = (45+25) - (6+4) = 60$$

Example

Evaluate $I = \int_C \{(x^2+ye^x)dx+(e^x+y)dy\}$ between A (0, 1) and B (1, 2).

As before, compare with $\int_C \{Pdx+Q\,dy\}$.

$P = \frac{\partial z}{\partial x} = x^2+ye^x \qquad \therefore\ z = \frac{x^3}{3} + ye^x + f(y)$

$Q = \frac{\partial z}{\partial y} = e^x+y \qquad \therefore\ z = ye^x + \frac{y^2}{2} + F(x)$

For these expressions to agree,

$f(y) = \frac{y^2}{2}$; $F(x) = \frac{x^3}{3}$ Then $I = \left[\frac{x^3}{3} + ye^x + \frac{y^2}{2}\right]_{(0,1)}^{(1,2)} = \frac{5}{6} + 2e$

So the main points are that, if (Pdx+Qdy) is an exact differential

(a) $I = \int_C (Pdx + Qdy)$ is independent of the path of integration

(b) $I = \oint_C (P\,dx + Q\,dy)$ is zero.

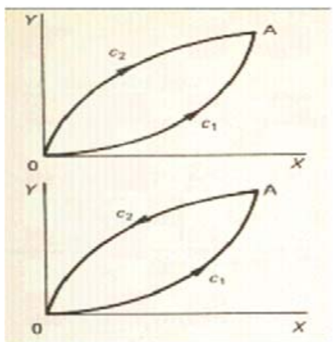
If $I = \int_C \{P\,dx + Q\,dy\}$ and (Pdx + Qdy) is an exact differential,

Then $I_{c_1} = I_{c_2}$

$I_{c_1} + I_{c_2} = 0$

Hence, the integration taken round a closed curve is zero, provided (Pdx+Q dy) is an exact differential.

$\therefore$ If (P dx + Q dy) is an exact differential, $\oint (P\,dx + Q\,dy) = 0$



Exact differentials in three independent variables

A line integral in space naturally involves three independent variables, but the method is very much like that for two independent variables.

$dz = Pdx + Q\,dy + R\,dw$ is an exact differential of $z = f(x, y, w)$

if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; $\frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}$; $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$

If the test is successful, then

(a) $\int_C (P\,dx + Q\,dy + R\,dw)$ is independent of the path of integration.

(b) $\oint_C (P\,dx + Q\,dy + R\,dw)$ is zero.

Example

Verify that $dz=(3x^2yw+6x)dx+(x^3w-8y)dy+(x^3y+1)\,dw$ is inexact differential and hence evaluate $\int_C dz$ from A (1, 2, 4) to B (2, 1 3).

First check that dz is an exact differential by finding the partial derivatives above, when $P = 3x^2yw + 6x$; $Q = x^3w - 8y$; and $R = x^3y + 1$

$$\frac{\partial P}{\partial y} = 3x^2w\,;\ \frac{\partial Q}{\partial x} = 3x^2w \quad \therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial w} = 3x^2y\,;\ \frac{\partial R}{\partial x} = 3x^2y \quad \therefore \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}$$

$$\frac{\partial R}{\partial y} = x^3;\ \frac{\partial Q}{\partial w} = x^3 \quad \therefore \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

$\therefore$ dz is an exact differential

Now to find z. $P = \frac{\partial z}{\partial x}$; $Q = \frac{\partial z}{\partial y}$; $R = \frac{\partial z}{\partial w}$

$$\therefore \frac{\partial z}{\partial x} = 3x^2yw+6x \ \therefore\ z=\int(3x^2yw+6x)dx \ = x^3yw+3x^2+f(y)+F(w)$$

$$\therefore \frac{\partial z}{\partial y} = x^3w-8x \ \therefore\ z=\int(x^3w-8y)dy \ = x^3yw-4y^2+g(x)+F(w)$$

$$\frac{\partial z}{\partial w} = x^3y+1 \quad \therefore\ z=\int(x^3y+1)dw \ = y^3yw+w+f(y)+g(x)$$

For these three expressions for z to agree

$f(y) = -4y^2$; $F(w) = w$; $g(x) = 3x^2$

$\therefore \quad z = x^3yw + 3x^2 - 4y^2 + w$

$$\therefore \quad I = \Big[x^3yw + 3x^2-4y^2+w\Big]_{(1,2,4)}^{(2,1,3)}$$

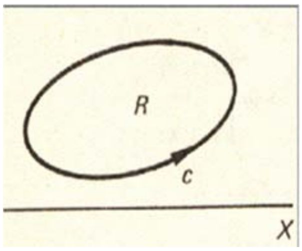
for $\quad I = \Big[x^3yw + 3x^2-4y^2+w\Big]_{(1,2,4)}^{(2,1,3)} = (24+12-4+3)-(8+3-16+4)=36$

The extension to line integrals in space is thus quite straightforward.

Finally, we have a theorem that can be very helpful on occasions and which links up with the work we have been doing. It is important, so let us start a new section.

Green's Thorem

Let P and Q be two function of x and y that are finite and continuous inside and the boundary c of a region R in the xy-plane.If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that.



$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\, dy = -\oint_C (P\, dx + Q\, dy)$$

That is, a double integral over the plane region R can be transformed into a line integral over the boundary c of the region – and the action is reversible.

Let us see how it works.

EXAMPLE

Evaluate $I = \oint_C \{(2x - y)dx + (2y+x)dy\}$ around the boundary c . the ellipse $x^2 + 9y^2 = 16$.

The integral is of the form

$I = \oint_C \{P\, dx + Q\, dy)$ where $P = 2x - y \therefore \dfrac{\partial P}{\partial y} = -1$ and $Q = 2y + x \therefore \dfrac{\partial Q}{\partial x} = 1$.

$$\therefore I = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx dy = -\iint_R (-1-1) dx\, dy = 2\iint_R dx\, dy$$

But $\iint_R dx\, dy$ over any closed region give the area of the figure.

In this case, then, I = 24 where A is the area of the ellipse

$x^2+9y^2 = 16$ i.e. $\dfrac{x^2}{16} + \dfrac{9y^2}{16} = 1$

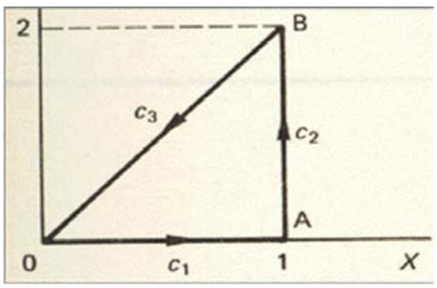
$\therefore\ a = 4;\ b = \dfrac{4}{3} \qquad \therefore\ A = \dfrac{16\pi}{3} \qquad \therefore\ I = 2A = \dfrac{32\pi}{3}$

To demonstrate the advantage of Green's theorem, let us work through the next example (a) by the previous method, and (b) by applying Green's theorem.

Example

Evaluate $I = \oint_C \{(2x+y)\, dx + (3x-2y)\, dy\}$ taken in anticlockwise manner round the triangle with vertices at O (0,0) A (1, 0) B (1, 2).

$$I = \oint_C \{(2x + y)\, dx + (3x - 2y)\, dy\}$$



(a) By the previous method

There are clearly three stages with c_1, c_2, c_3. Work through the complete evaluation to determine the value of I. It will be good revision. When you have finished, check the result with the solution in the next frame. $I = 2$

(a) (i) c_1 is $y = 0 \quad \therefore \; dy = 0$

$$\therefore \; I_1 = \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1 \quad \therefore \; I_1 = 1$$

(ii) c_2 is $x = 1 \quad \therefore \; dx = 0$

$$\therefore \; I_2 = \int_0^2 (3-2y)\,dy = \left[3y - y^2\right]_1^0 = 2 \quad \therefore \; I_2 = 2$$

(iii) c_3 is $y = 2x \quad \therefore \; dy = 2\,dx$

$$\therefore \; I_3 = \int_1^0 \{4x\,dx + (3x - 4x)\,2\,dx\}$$

$$= \int_1^0 2x\,dx = \left[x^2\right]_1^0 = -1 \quad \therefore \; I_3 = -1$$

$$I = I_1 + I_2 + I_3 = 1 + 2 + (-1) = 2 \quad \therefore \; I = 2$$

Now we will do the same problem by applying Green's theorem, so more

(b) By Green's theorem

$$I = \oint_C \{(2x + y)\,dx + (3x - 2y)\,dy\}$$

$$P = 2x + y \quad \therefore \quad \frac{\partial P}{\partial y} = 1;$$

$$Q = 3x - 2y \quad \therefore \quad \frac{\partial Q}{\partial x} = 3$$

$$I = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

Finish it off. $I = 2$

$$\text{For } I = -\iint_R (1-3)\,dx\,dy = 2\iint_R dx\,dy = 2A$$

$$= 2 \times \text{the area of the triangle} = 2 \times 1 = 2$$

$\therefore \; I = 2$

Application of Green's theorem is not always the quickest method. It is useful, however, to have both methods available.

If you have not already done so, make a note of Green's theorem.

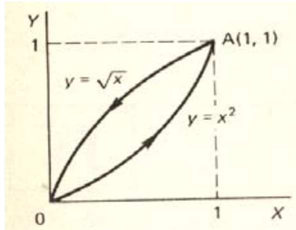
$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = -\oint_C (P\,dx + Q\,dy)$$

Note: Green's theorem can, in fact, be applied to a region that is not simply connected by arranging a link between outer and inner boundaries, provided the path of integration is such that the region is kept on the left-hand side.

Lecture -34………Examples

Example

Evaluate the line integral $I = \oint_C \{xy\,dx + (2x - y)\,dy\}$ round the region bounded by the curves $y = x^2$ and $x = y^2$ by the use of Green's theorem.



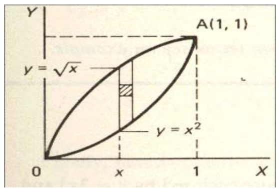
Points of intersection are O(0, 0) and A(1, 1).

$$I = \oint_C \{xy\,dx + (2x - y)\,dy\}$$

$$\oint_C \{P dx + Q dy\} = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$P = xy \quad \therefore \frac{\partial P}{\partial y} = x; \quad Q = 2x - y \quad \therefore \frac{\partial Q}{\partial x} = 2$$

$$I = -\iint_R (x - 2)\,dx\,dy = -\int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x - 2)\,dy\,dx$$



$$= -\int_0^1 (x - 2)\,\Big[\,y\,\Big]_{x^2}^{\sqrt{x}}\,dx$$

$$\therefore\ I = -\int_0^1 (x - 2)\left(\sqrt{x} - x^2\right)\,dx = -\int_0^1 (x^{3/2} - x^3 - 2x^{1/2} + 2x^2)\,dx$$

$$= -\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 - \frac{4}{3}x^{3/2} + \frac{2}{3}x^3\right]_0^1 = \frac{31}{60}$$

In this special case when P=y and Q= − x so $\frac{\partial P}{\partial y} = 1$ and $\frac{\partial Q}{\partial x} = -1$

Green's theorem then states $\iint_R \{1 - (-1)\}\,dx\,dy = -\oint_C (P\,dx + Q\,dy)$

i.e. $\quad 2\iint_R dx\,dy = -\oint_C (y\,dx - x\,dy) = \oint_C (x\,dy - y\,dx)$

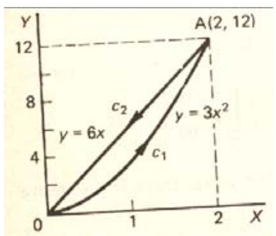
Therefore, the area of the closed region $A = \iint_R dx\,dy = \frac{1}{2}\oint_C (x\,dy - y\,dx)$

Example

Determine the area of the figure enclosed by $y = 3x^2$ and $y = 6x$.

Points of intersection : $3x^2 = 6x \quad \therefore\ x = 0$ or 2

$$\text{Area } A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$



We evaluate the integral in two parts, i.e.
OA along c_1 and AO along c_2

$$2A = \int_{c_1 \text{ (along OA)}} (x dy - y dx) + \int_{c_2 \text{ (along OA)}} (x dy - y dx) = I_1 + I_2$$

I_1: c_1 is $y = 3x^2 \quad \therefore\ dy = 6x\,dx$

$$\therefore\ I_1 = \int_0^2 (6x^2dx - 3x^2dx) = \int_0^2 3x^2\,dx = \left[\,x^3\,\right]_0^2 = 8 \ \therefore\ I_1 = 8$$

Similarly, for c_2 is $y = 6x$ $\quad\therefore$ $dy = 6\,dx$

$$\therefore\ I_2 = \int_2^0 (6x\,dx - 6x\,dx) = 0$$

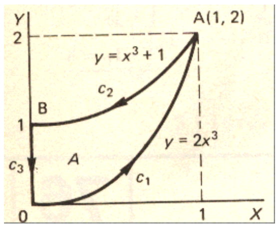
$\therefore\ I_2 = 0$

$\therefore\ I = I_1 + I_2 = 8 + 0 = 8$

$\therefore\ A = 4$ square units

Example

Determine the area bounded by the curves $y = 2x^3$, $y = x^3 + 1$ and the axis $x = 0$ for $x \geq 0$.



Here it is $y = 2x^3$; $y = x^3 + 1$; $x = 0$

Point of intersection $2x^3 = x^3 + 1$ $\therefore$ $x^3 = 1$ $\therefore$ $x = 1$

Area $A = \dfrac{1}{2}\oint_C (x\,dy - y\,dx)$ $\therefore$ $2A = \oint_C (x\,dy - y\,dx)$

(a) OA : c_1 is $y = 2x^3$ $\therefore$ $dy = 6x^2\,dx$

$$\therefore I_1 = \int_{c_1} (xdy - ydx) = \int_0^1 (6x^3dx - 2x^3\,dx) \quad = \int_0^1 4x3\,dx = \left[\,x^4\,\right]_0^1 = 1$$

$\therefore\ I_1 = 1$

(b) AB: c_2 is $y = x^3 + 1$ $\therefore$ $dy = 3x^2\,dx$

$$\therefore\ I_2 = \int_1^0 \{3x^3\,dx - (x^3 + 1)\,dx\} = \int_1^0 (2x^3 - 1)\,dx = \left[\frac{x^4}{2} - x\right]_1^0 = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

$$\therefore\ I_2 = \frac{1}{2}$$

(c) BO: c_3 is $x = 0$ $\therefore$ $dx = 0$

$$I_3 = \int_{y=1}^{y=0} (xdy - ydx) = 0 \quad \therefore\ I_3 = 0$$

$$\therefore\ 2A = I = I_1 + I_2 + I_3 = 1 + \frac{1}{2} + 0 = 1\frac{1}{2} \qquad \therefore\ A = \frac{3}{4} \text{ square units}$$

Revision Summary

Properties of line integrals

- Sign of line integral is reversed when the direction of integration along the path is reversed.
- Path of integration parallel to y-axis, $dx = 0$ $\therefore$ $I_c = \int_c Q\,dy$.
- Path of integration parallel to x-axis, $dy = 0$ $\therefore$ $I_c = \int_c P\,dx$.
- Path of integration must be continuous and single-valued.
- Dependence of line integral on path of integration.
- In general, the value of the line integral depends on the particular path of integration.
- Exact differential

  If $P\,dx + Q\,dy$ is an exact differential

(a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(b) $I = \int_c (P\,dx + Q\,dy)$ is independent of the path of integration

(c) $I = \oint_C (P\,dx + Q\,dy)$ is zero.

- Exact differential in three variables.
  If P dx + Q dy + R dw is an exact differential
  (a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; $\frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}$; $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$
  (b) $\int_c (P\,dx + Q\,dy + R\,dw)$ is independent of the path of integration.
  (c) $\oint_C (P\,dx + Q\,dy + R\,dw)$ is zero.
- Green's theorem
  $\oint_C (P\,dx+Q\,dy) = -\iint_R \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} dx\,dy$ and, for a simple closed curve,
  $\oint_C (x\,dy - y\,dx) = 2\iint_R dx\,dy = 2A$
  where A is the area of the enclosed figure.

Gradient of a scalar function

Del operator is given by $\nabla = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$

$\nabla\phi = \text{grad}\,\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$

$\text{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

Div (Divergence of a vector function)

If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

then $\text{div}\,A = \nabla.A = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)$

$\therefore\ \text{div}\,\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$

Note that

(a) the grad operator ∇ acts on a scalar and gives a vector

(b) the div operator $\nabla.$ acts on a vector and gives a scalar.

Example

If $A = x^2y\mathbf{i} - xyz\mathbf{j} + yz^2\mathbf{k}$ then

Div $\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(yz^4) = 2xy - xz + 2yz$

Example

If A = $2x^2y\mathbf{i} - 2(xy^2+y^3)\mathbf{j}+3y^2z^2\mathbf{k}$ determine $\nabla.\mathbf{A}$ i.e. div **A**.

A= $2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$

$$\nabla.\mathbf{A}= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 4xy-2(2xy+3y^2z)+6y^2z= 4xy - 4xy - 6y^2z+6y^2z = 0$$

Such a vector A for which $\nabla.\mathbf{A} = 0$ at all points, i.e. for all values of x, y, z, is called a solenoid vector. It is rather a special case.

Curl (curl of a vector function)

The curl operator denoted by $\nabla\times$, acts on a vector and gives another vector as a result.

If $\mathbf{A}= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then curl $\mathbf{A}=\nabla\times\mathbf{A}$.

i.e. curl $\mathbf{A}=\nabla\times\mathbf{A}= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}+ \mathbf{k}\frac{\partial}{\partial z}\right)\times \left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\therefore\ \nabla\times\mathbf{A}= \mathbf{i}\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) +\mathbf{j}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

Curl A is thus a vector function.

Example

If $\mathbf{A}=(y^4-x^2z^2)\mathbf{i}+(x^2+y^2)\mathbf{j}-x^2yzk$,determine curl A at the point (1,3, –2).

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y4-x^2z^2 & x^2+y^2 & -x^2yz \end{vmatrix}$$

Now we expand the determinant

$$\nabla \times \mathbf{A}= \mathbf{i}\left\{\frac{\partial}{\partial y}(- x^2yz) - \frac{\partial}{\partial z}(x^2+y^2)\right\} - \mathbf{j}\left\{\frac{\partial}{\partial x}(- x^2yz) - \frac{\partial}{\partial z}(y^4 - x^2z^2)\right\}$$

$$+ \mathbf{k}\left\{\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2z^2)\right\}$$

$\nabla\times\mathbf{A}=\mathbf{i}\{-x^2z\}-\mathbf{j}\{-2xyz+2x^2z\}+\mathbf{k}(2x-4y^3)$. $\therefore$ At (1, 3, – 2),

$\nabla\times\mathbf{A} = \mathbf{i}\,(2)-\mathbf{j}\,(12 - 4) + \mathbf{k}\,(2 - 108) \quad = 2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}$

Example

Determine curl F at the point (2,0,3) given that $\mathbf{F}=ze^{2xy}\mathbf{i}+2xycosy\mathbf{j}+(x+2y)\mathbf{k}$.

$$\text{In determine form, curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{2xy} & 2xzcosy & x+2y \end{vmatrix}$$

Now expand the determinant and substitute the values for x, y and z, finally obtaining curl

$\nabla \times \mathbf{F} = \mathbf{i}\{2 - 2x \cos y\}- \mathbf{j}\{1 - e^{2xy}\}+ \mathbf{k}\,(\{2z \cos y - 2xze^{2xy}\}$

$\therefore$At(2,0,3) $\nabla\times\mathbf{F}=\mathbf{i}(2-4)-\mathbf{j}(1-1)+\mathbf{k}(6-12) = -2\mathbf{i}-6\mathbf{k} = -2\,(\mathbf{i} + 3k)$

Summary of grad, div and curl

(a) Grad operator ∇ acts on a scalar field to give a vector field.
(b) Div operator ∇**.** Acts o a vector field to give a scalar field.
(c) Curl operator $\nabla \times$ acts on a vector field to give a vector field.
(d) With a scalar function ϕ (x,y,z)

$$\text{Grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

(e) With a vector function $A = a_x i + a_y \mathbf{j} + a_z \mathbf{k}$

(i) $\text{div } \mathbf{A} = \nabla.\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

(ii) $\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$

Multiple Operations

We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

EXAMPLE

If $A = x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}$

Then $\text{div } \mathbf{A} = \nabla.\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).(x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k})$

$= 2xy + z^3 + x^3 = \phi$ (say)

Then $\text{grad (div } \mathbf{A}) = \nabla(\nabla.\mathbf{A}) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (2y+3x^2)\mathbf{i} + (2x)\mathbf{j} + (3z^2)\mathbf{k}$

i.e., $\text{grad div } A = \nabla(\nabla.\mathbf{A}) = (2y+3x^2)\mathbf{i} + 2x\mathbf{j} + 3z^2\mathbf{k}$

Example

If $\phi = xyz - 2y^2z + x^2z^2$ determine div grad ϕ at the point (2, 4, 1).
First find grad ϕ and then the div of the result.
$\text{div grad } \phi = \nabla.(\nabla\phi)$
We have $\phi = xyz - 2y^2z + x^2z^2$

$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (yz+2xz^2)\mathbf{i} + (xz-4yz)\mathbf{j} + (xy-2y^2+2x^2z)\mathbf{k}$

$\therefore \text{div grad } \phi = \nabla.(\nabla\phi) = 2z^2 - 4z + 2x^2$

$\therefore$ At (2,4,1), $\text{div grad } \phi = \nabla.(\nabla\phi) = 2 - 4 + 8 = 6$

$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

Then $\text{div grad } \phi = \nabla.(\nabla\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}$

$\therefore \text{div grad } \phi = \nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

Example

If F = $x^2yz\mathbf{i} + xyz^2\mathbf{j} + y^2z\mathbf{k}$ determine curl **F** at the point (2, 1, 1).Determine an expression for curl F in the usual way, which will be a vector, and then the curl of the result. Finally substitute values.

Curl curl $\mathbf{F}=\nabla\times(\nabla\times\mathbf{F}) = \mathbf{i}+2\mathbf{j}+6\mathbf{k}$

$$\text{For curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix} = (2yz-2xyz)\mathbf{i}+x^2y\mathbf{j}+(yz^2-x^2z)\mathbf{k}$$

$$\text{Then Curl Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz-2xyz & x^2y & yz^2-x^2z \end{vmatrix} = z^2i-(-2xz-2y+2xy)\mathbf{j}+(2xy-2z+2xz)\mathbf{k}$$

$\therefore$ At (2, 1, 1), curl cul $\mathbf{F}=\nabla\times(\nabla\times\mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

Two interesting general results

(a) Curl grad ϕ where ϕ is a scalar

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\therefore \text{ curl grad } \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right\} - \mathbf{j}\left\{\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right\} + \mathbf{k}\left\{\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right\} = 0$$

$\therefore$ curl grad $\phi = \nabla \times (\nabla\phi) = 0$

(b) Div curl A where A is a vector.

$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\text{Then div curl } \mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \mathbf{.} (\nabla \times \mathbf{A})$$

$$= \frac{\partial^2 a_z}{\partial x\partial y} - \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_z}{\partial x\partial y} + \frac{\partial^2 a_x}{\partial y\partial z} - \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_x}{\partial y\partial z} = 0$$

$\therefore$ div curl $\mathbf{A} = \nabla \mathbf{.} (\nabla \times \mathbf{A}) = 0$

(c) Div grad ϕ where ϕ is a scalar.

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\text{Then div grad } \phi = \nabla.(\nabla\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\therefore \text{div grad } \phi = \nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

This result is sometimes denoted by $\nabla^2\phi$.

So these general results are

(a) curl grad $\phi = \nabla\times(\nabla\phi) = 0$
(b) div curl $\mathbf{A} = \nabla\mathbf{.}(\nabla\times\mathbf{A}) = 0$
(c) div grad $\phi=\nabla.(\nabla\phi)=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

Lecture No -35 Definite Integrals

Definite integral for $\sin^n x$ and $\cos^n x$, $0 \le x \le \pi/2$

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)dx = \frac{1}{2}\left| x - \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}\left|\frac{\pi}{2} - \frac{\sin\pi}{2}\right| = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)dx = \frac{1}{2}\left| x + \frac{\sin 2x}{2}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}\left|\frac{\pi}{2} + \frac{\sin\pi}{2}\right| = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x dx = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx = \int_0^{\frac{\pi}{2}} \sin x dx + \int_0^{\frac{\pi}{2}} \cos^2 x(-\sin x)dx$$

$$= \left|-\cos x\right|_0^{\frac{\pi}{2}} + \left|\frac{\cos^3 x}{3}\right|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 + \frac{1}{3}\left[\cos^3\frac{\pi}{2} - \cos^3 0\right] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos^2 x \cos x dx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cos x dx = \int_0^{\frac{\pi}{2}} \cos x dx - \int_0^{\frac{\pi}{2}} \sin^2 x(\cos x)dx$$

$$= \left|\sin x\right|_0^{\frac{\pi}{2}} - \left|\frac{\sin^3 x}{3}\right|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 - \frac{1}{3}\left[\sin^3\frac{\pi}{2} - \sin^3 0\right] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \int_0^{\frac{\pi}{2}} (\sin^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[\frac{1-\cos 2x}{2}\right]^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2x + \cos^2 2x)dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2x + \frac{1+\cos 4x}{2})dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\frac{3}{2} - 2\cos 2x + \frac{c\,os 4x}{2})dx$$

$$= \frac{1}{4}\left|\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right|_0^{\frac{\pi}{2}} = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2} - \sin\pi + \frac{\sin 2\pi}{8}\right]$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2}\right] \qquad so \qquad \int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx = \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[\frac{1+\cos 2x}{2}\right]^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1+2\cos 2x+\cos^2 2x)dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1+2\cos 2x+\frac{1+\cos 4x}{2})dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\frac{3}{2}+2\cos 2x+\frac{c\text{os}\, 4x}{2})dx$$

$$= \frac{1}{4}\left|\frac{3}{2}x+\sin 2x+\frac{\sin 4x}{8}\right|_0^{\frac{\pi}{2}} = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2}+\sin\pi+\frac{\sin 2\pi}{8}\right]$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2}\right] \qquad \text{So} \qquad \int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^5 x dx = \frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^5 x dx = \frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^6 x dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{6}{7}\frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^7 x dx = \frac{6}{7}\frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^9 x dx = \frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^9 x dx = \frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

Wallis Sine Formula

When n is even $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\frac{n-7}{n-6}.--------\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$

When n is odd $\displaystyle\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\frac{n-7}{n-6}.--------\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.$

$$\int_0^{\frac{\pi}{2}} \sin^{11} x dx = \frac{10.8.6.4.2}{11.9.7.5.3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{11} x dx = \frac{10.8.6.4.2}{11.9.7.5.3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{12} x dx = \frac{11.9.7.5.3.1}{10.8.6.4.2}\frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{12} x dx = \frac{11.9.7.5.3.1}{10.8.6.4.2}\frac{\pi}{2}$$

Integration By Parts

$$\int UVdx = U\int V\,dx - \int\left[\int Vdx.\frac{dU}{dx}\right]dx$$

Example *Evaluate* $\int$ x lnx dx

$\int$ x lnx dx= lnx $\int$ xdx – $\int$ [$\int$ x dx.$\frac{d}{dx}$(ln x)] dx (We are integrating by parts)

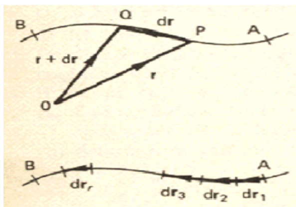
$= \ln x(\frac{x^2}{2}) - \int (\frac{x^2}{2})(\frac{1}{x})dx = (\frac{x^2}{2}) \ln x - \int (\frac{x}{2})d\,x = (\frac{x^2}{2}) \ln x - \frac{1}{2}(\frac{x^2}{2})$

Example *Evaluate* $\int$ x sinx dx

$\int$ **x** sinx dx =x$\int$ sin xdx – $\int$ [$\int$ sinx dx.$\frac{d}{dx}$(x)] dx (We are integrating by parts)

= x(-cosx)-$\int$ (-cosx)(1)dx= -x(cosx) +$\int$ cosx d x = -x(cosx)+sin x

Line Integrals

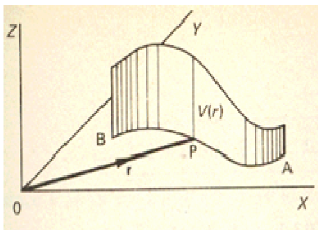


Let a point p on the curve c joining A and B be denoted by the position vector **r** with respect to origin O. If q is a neighboring point on the curve with position vector **r+dr** , then $\overline{PQ}$ = **r**

The curve c can be divided up into many n such small arcs , approximating to d**r**$_1$, d**r**$_2$, d**r**$_3$,………. d**r**$_p$,……

so that $\overline{AB}\sum_{p=1}^{n} dr_p$ where d**r**$_p$ is a vector representing the element of the arc in both magnitude and direction. If dr $\to$ 0 , then the length of the curve AB=$\int_c dr$.

Scalar Field



If a scalar field V(r) exists for all points on the curve ,

the $\sum_{p=1}^{n} V(r)dr_p$ with dr $\to$ 0 , defines the line integral

of V i.e line integral = $\int_c V(r)dr.$

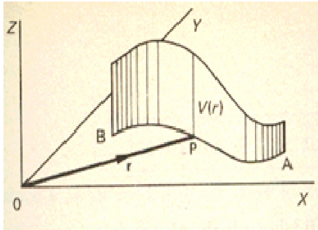
We can illustrate this integral by erecting a continuous Ordinate to V(r) at each point of the curve $\int_c V(r)dr$ is then represented by the area of the curved surface between the ends A and B the curve c. To evaluate a line integral , the integrand is expressed in terms of x , y ,z with dr =dx + dy j + dz k

In practice , x , y and are often expressed in terms of parametric equation of a fourth variable (say u), i.e. x = x(u) ; y = y(u) ; z = z(u) . From these , dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.

Lecture No -36 Scalar Field

Scalar Field

If a scalar field V(r) exists for all points on the curve ,



the $\sum_{p=1}^{n} V(r)dr_p$ with dr $\rightarrow 0$, defines the line integral of V i.e line integral = $\int_c V(r)dr.$

We can illustrate this integral by erecting a continuous Ordinate to V(r) at each point of the curve $\int_c V(r)dr$ is then represented by the area of the curved surface between the ends A and B the curve c. To evaluate a line integral , the integrand is expressed in terms of x , y ,z with dr =dx + dy j + dz k

In practice , x , y and are often expressed in terms of parametric equation of a fourth variable (say u), i.e. x = x(u) ; y = y(u) ; z = z(u) . From these , dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.

Example

If $V=xy^2z$, evaluate $\int_c V(r)dr$ along the curve c having parametric equations

$x = 3u; y=2u^2 ;z=u^3$ between A(0,0,0) and B(3,2,1)

$V= xy^2z = (3u)(4u^4)(u^3)=12u^8$

$d\mathbf{r}= dx\mathbf{i}+ dy\,\mathbf{j}+ dz\,\mathbf{k} \Rightarrow \mathbf{dr= 3}du\ i +4udu\ j +\mathbf{3u^2}du\ k$

for $x = 3u$; $\therefore$ $dx = 3du$; $y = 2u^2$ $\therefore$ $dy = 4u\,du$; $z = u^3$ $\therefore$ $dz =\mathbf{3u^2dz}$

Limiting : A(0,0,0) corresponds to B(3,2,1) corresponds to u

A(0,0,0) $\equiv$ u=0 ; B(3,2,1) $\equiv$ u = 1

$$\int_c V(r)dr = \int_0^1 12u^8(3\ i +4u\ j +3u^2\ k)du = \left| 36\frac{u^9}{9}i + 48\frac{u^{10}}{10}j + 36\frac{u^{11}}{11} \right|_0^1 = 4i + \frac{24}{5}j + \frac{36}{11}k$$

Example

If $V = xy + y^2z$ Evaluate $\int_c V(r)dr$ along the curve c defined by $x= t^2$; $y = 2t$; $z= t+5$ between A(0,0,5) and B(4,4,7) . As before , expressing V and d**r** in term of the parameter t .

since $V=xy+y^2z$

$= (t^2)(2t)+(4t^2)(t+5)$

$= 6t^3 + 20t^2.$

$$\left.\begin{array}{ll} x = t^2 & dx = 2t\,dt \\ y = 2t & dy = 2\,dt \\ z = t+5 & dz = dt \end{array}\right\}$$

$\therefore$ $dr = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$

$= 2t\,dt\ \mathbf{i} + 2\,dt\,\mathbf{j} + dt\mathbf{k}$

$\therefore \int_C Vdr = \int_C (6t^3+20t^2)(\,2t\ \mathbf{i} +2\mathbf{j}+\mathbf{k})\,dt$

Limits: A (0, 0, 5) $\equiv t = 0$;
B (4, 4, 7) $\equiv t = 2$

$$\therefore \int_C V dr = \int_0^2 (6t^3+20t^2)(2t\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k})\,dt$$

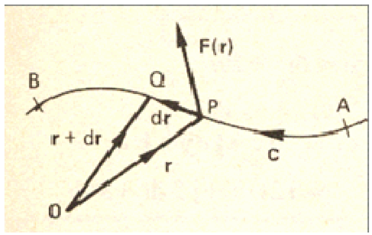
$$\int_C V dr = 2\int_0^2 \{(6t^4+20t^3)\mathbf{i}+(6t^3+20t^2)\mathbf{j}$$
$$+(3t^3+10t^2)k\}dt.$$
$$= \frac{8}{15}(444\mathbf{i} + 290\mathbf{j} + 145\mathbf{k})$$

Vector Field

If a vector field F(r) exist for all points of the curve c, then for each element of arc we can form the scalar product F.dr. Summing these products for all elements of arc, we have

$$\sum_{p=1}^{n} F.dr_p$$

The line integral of F(r) from A to B along the stated curve = $\int_C$ F.dr.



In this case, since **F**.d**r** is a scalar product, then the line integral is a scalar. To evaluate the line integral, F and d**r** are expressed in terms of x,y,z, and the curve in parametric form. We have

$\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$

And $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

Then $\mathbf{F}.d\mathbf{r} = (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}).(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \int_c (F_1 dx + F_2 dy + F_3 dz)$

Now for an example to show it in operation.

Example

If $F(r) = x^2y\,\mathbf{i} + xz\,\mathbf{j} + 2yz\,\mathbf{k}$, Evaluate $\int_c F.dr$ between A(0,0,0) and B(4,2,1) along the curve c having parametric equations $x=4t$; $y-2t^2$; $z = t^3$

Expressing everthing in terms of the parameter t, we have

$dx = 4\,dt$; $dy = 4t\,dt$; $dz = 3t^2\,dt$

$x^2y = (16t^2)(2t^2) = 32\,t^4$

$x = 4t \quad \therefore \quad dx = 4\,dt$

$xz = (4t)(t^3) = 4\,t^4$

$y = 2\,t^2 \quad dy = 4t\,dt$

$2yz = (4\,t^2)(t^3) = 4\,t^5$

$z = t^3 \quad \therefore\ dz = 3t^2\,dt$

$\mathbf{F} = 32\,t^4\,\mathbf{i} + 4\,t^4\,\mathbf{j} - 4\,t^5\,\mathbf{k}$

$dr = 4dt\,\mathbf{i} + 4t\,dt\,\mathbf{j} + 3t^2\,\mathbf{k}$

Then $\int \mathbf{F}.d\mathbf{r} = \int (32t^4\mathbf{i}+4t^4\mathbf{j}-4t^5\mathbf{k}).$
$(4dt\,\mathbf{i} + 4t\,dt\,\mathbf{j} + 3t^2\,dt\,\mathbf{k})$

$= \int (128t^4+16t^5+12t^7)\,dt$

Limits: A(0,0,0) $\equiv t = 0$;
B (4, 2, 1) $\equiv t = 1$

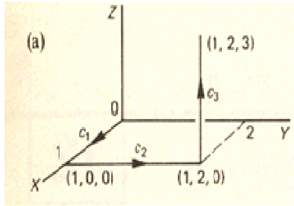
$$\int_C \mathbf{F}.d\mathbf{r} = (128t^4 + 16t^5 + 12t^7)dt = \frac{128}{5}t^5 + \frac{16}{6}t^6 + \frac{12}{8}t^8 = \frac{128}{5} + \frac{8}{3} + \frac{3}{2} = 29.76$$

Example

If $\mathbf{F}(\mathbf{r}) = x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}$

Evaluate $\int_C \mathbf{F.dr}$ between A(0,0,0) and B(1,2,3)

B (1, 2, 3)



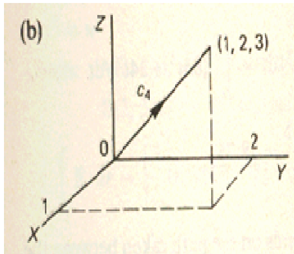
(a) along the straight line

c_1 from (0, 0, 0) to (1, 0, 0)

then c_2 from (1, 0, 0) to (1, 2, 0)

and c_3 from (1, 2, 0) to (1, 2, 3)

(b) along the straight line c_4 joining (0, 0, 0) to (1, 2, 3).



We first obtain an expression for **F.dr** which is

$\mathbf{F.dr} = (x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}).(dx\,\mathbf{i} + dy\mathbf{j} + dz\,\mathbf{k})$

$\mathbf{F.dr} = x^2y\,dx + 2yz\,dy + 3z^2x\,dz$

$\int \mathbf{F.dr} = \int x^2ydx + \int 2yzdy + \int 3z^2xdz$

Here the integration is made in three sections, along c_1, c_2 and c_3.

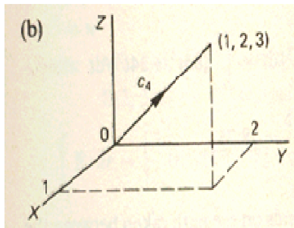
(i) c_1: $y = 0, z = 0, dy = 0, dz = 0$

$\therefore \int_{c_1} \mathbf{F.dr} = 0+0+0 = 0$

(ii) c_2: The conditions along c_2 are

c_2: $x = 1, z = 0, dx = 0, dz = 0$

$\therefore \int_{c_2} F.dr = 0 + 0 + 0 = 0$



(iii) c_3: $x = 1, y = 2, dx = 0, dy = 0$

$\int_{c_3} F.dr = 0 + 0 + \int_0^3 3z^2dz = 27$

Summing the three partial results

$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F.dr} = 0 + 0 + 27 = 27$

$\therefore \int_{c_1+c_2+c_3} \mathbf{F.dr} = 27$

If t taken as the parameter, the parametric equation of c are $x = t; y = 2t; z = 3t$

$(0, 0, 0) \Rightarrow t = 0, (1, 2, 3) \Rightarrow t = 1$ and the limits of t are $t = 0$ and $t = 1$

$\mathbf{F} = 2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}$

$\mathbf{dr} = dx\mathbf{i} + dy\mathbf{j} + \mathbf{k}dz = dt\,\mathbf{i} + 2dt\,\mathbf{j} + 3dt\,\mathbf{k}$

$\int_{c_4} \mathbf{F.dr} = \int_0 (2t^3\mathbf{i}+12t^3\mathbf{j}+27t^3\mathbf{k}).(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\,dt = \int_0^1 (2t^3+24t^2+81t^3)dt$

$= \int_0^1 (83t^3 + 24t^2)\,dt = \left[83\frac{t^4}{4} + 8t^3\right]_0^1 = \frac{115}{4} = 28.75$

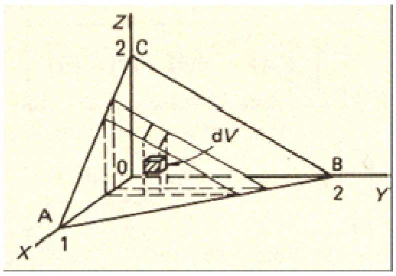
So the value of the line integral depends on the path taken between the two end points A and B

(a) $\int \mathbf{F.dr}$ via c_1, c_2 and c_3 = 27

(b) $\int \mathbf{F.dr}$ via c_4 = 28.75

Example

Evaluate $\int_v F\,dv$ where V is the region bounded by the planes x = 0 ,y = 0 , z = 0 and 2x + y = 2 , and F = 2z **i** +y **k.** To sketch the surface 2x + y + z = 2, note that



when z = 0, 2x+y=2 i.e. y = 2 − 2x
when y = 0, 2x+z=2 i.e. z = 2 − 2x
when x = 0, y+z=2 i.e. z = 2 − y
Inserting these in the planes
x = 0, y = 0, z = 0 will help.
The diagram is therefore.

So 2x + y + z = 2 cuts the axes at
A(1,0,0); B (0, 2, 0); C (0, 0, 2).
Also F = 2z**i** + y**k**;
z = 2 − 2x − y = 2 (1 − x) − y

$$\therefore \int_v F dV = \int_0^1 \int_0^{2(1-x)} \int_0^{2(1-x)-y} (2x\mathbf{i}+y\mathbf{k})dzdydx$$

$$= \int_0^1 \int_0^{2(1-x)} \left[z^2\mathbf{i}+yz\mathbf{k}\right]_{z=0}^{z=2(1-x)-y} dydx$$

$$= \int_0^1 \int_0^{2(1-x)} \{[4(1-x)^2-4(1-x)y+y^2]\mathbf{i} + [2(1-x)y-y^2]k\} dydx$$

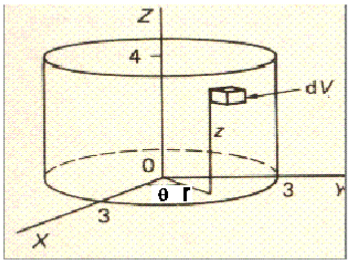
$$\int_v FdV = \frac{1}{3}(2\mathbf{i}+\mathbf{k})$$

Lecture No -37 Examples

Example

Evaluate $\int_v \mathbf{F}dV$ where $\mathbf{F}=2\mathbf{i}+2z\mathbf{j}+y\mathbf{k}$ and V is the region bounded by the planes z = 0, z = 4 and the surface

$x^2+y^2 = 9$.



It will be convenient to use cylindrical polar coordinates (r, θ, z) so the relevant transformations are

$x = r\cos\theta$; $y = r\sin\theta$ $z = z$; $dV=r\,dr\,d\theta\,dz$

Then $\int_v FdV= \iiint_v(2i+2zj+yk)dxdydz$

Changing into cylindrical polar coordinates with appropriate change of limits this becomes

$$\int_v FdV=\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{4}(2i+2zj+r\sin\theta k)rdzdrd\theta=\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\Big[2z\mathbf{i}+z^2\mathbf{j}+r\sin\theta z\mathbf{k}\Big]_{z=0}^{4} rdrd\theta$$

$$=\int_0^{2\pi}\int_0^{3}(8\mathbf{i}+16\mathbf{j}+4r\sin\theta\mathbf{k})rdrd\theta= 4\int_0^{2\pi}\int_0^{3}(2r\mathbf{i}+4r\mathbf{j}+r^2\sin\theta\mathbf{k})\,drd\theta=4\int_0^{2\pi}\left[r^2\mathbf{i}+2r^2\mathbf{j}+\frac{r^3}{3}\sin\theta\,\mathbf{k}\right]_0^3 d\theta$$

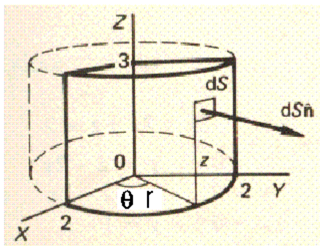
$$= 4\int_0^{2\pi}(9\mathbf{i}+18\mathbf{j} + 9\sin\theta\mathbf{k})\,d\theta= 36\int_0^{2\pi}(\mathbf{i}+2\mathbf{j}+\sin\theta\mathbf{k})d\theta= 36\Big[\theta\mathbf{i} + 2\theta\mathbf{j} - \cos\theta\mathbf{k}\Big]_0^{2\pi}$$

$$= 36\ \{(2\pi\mathbf{i} + 4\pi\mathbf{j} - \mathbf{k})-(-k)\}= 72\pi\ (\mathbf{i} + 2\mathbf{j})$$

Scalar Fields

A scalar field F = xyz exists over the curved surface S defined by $x^2+y^2= 4$ between the planes z = 0 and z = 3 in the first octant.

Evaluate $\int_S$ F dS over this surface.



We have F = xyz S: $x^2+y^2-4 = 0$, z = 0 to z = 3

$$d\mathbf{S} = \hat{\mathbf{n}}\,dS \quad \text{where } \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

$$\text{Now } \nabla S = \frac{\partial S}{\partial x}\mathbf{i}+\frac{\partial S}{\partial y}\mathbf{j}+\frac{\partial S}{\partial z}k = 2x\mathbf{i}+2y\mathbf{j}$$

$$|\nabla S| = \sqrt{4x^2+4y^2} =2\sqrt{x^2+y^2} =2\sqrt{4} = 4$$

$$\therefore\ \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{x\mathbf{i} + y\mathbf{j}}{2}$$

$$\therefore\ d\mathbf{S} = \hat{\mathbf{n}}dS = \frac{x\mathbf{i}+y\mathbf{j}}{2}dS$$

$$\therefore \int_S Fd\mathbf{S}=\int_S F\hat{\mathbf{n}}dS= \frac{1}{2}\int_S xyz(x\mathbf{i}+y\mathbf{j})dS = \frac{1}{2}\int_S (x^2yz\mathbf{i}+xy^2z\mathbf{j})dS \quad (1)$$

We have to evaluate this integral over the prescribed surface.

Changing to cylindrical coordinates with r = 2

$x = 2\cos\theta$; $y = 2\sin\theta z = z$; $dS = 2d\theta dz$

$\therefore\ x^2yz = (4\cos^2\theta)(2\sin\theta)(z)= 8\cos^2\theta\sin\theta z$

$xy^2z = (2\cos\theta)(4\sin^2\theta)(z)=8\cos\theta\sin^2\theta z$

Then result $\int_S F d\mathbf{S} = \frac{1}{2}\int_S (x^2yz\mathbf{i}+xy^2z\mathbf{j})dS$ becomes.

$$\int_S F dS = \frac{8}{2}\int_0^{\pi/2}\int_0^{3} (\cos^2\theta\sin\theta\mathbf{i}+\cos\theta\sin 2\theta\mathbf{j})2z dz d\theta = 4\int_0^{\pi/2}\int_0^{3}(\cos^2\theta\sin\theta\mathbf{i}+\cos\theta\sin^2\theta\mathbf{j})2z dz\, d\theta$$

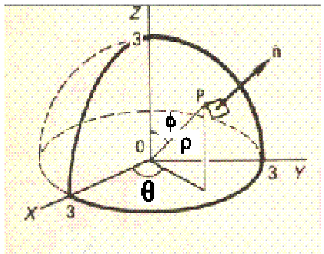
$$= 4\int_0^{\pi/2} (\cos^2\theta\sin\theta\mathbf{i}+\cos\theta\sin^2\theta\mathbf{j})z^2\Big|_0^3 d\theta = 4\int_0^{\pi/2} (\cos^2\theta\sin\theta\mathbf{i}+\cos\theta\sin^2\theta\mathbf{j})9 d\theta$$

$$= 36\left[-\frac{\cos^3\theta}{3}\mathbf{i} + \frac{\sin^3\theta}{3}\mathbf{j}\right]_0^{\pi/2} = 12\,(\mathbf{i}+\mathbf{j})$$

Vector Field

A vector field $\mathbf{F}=y\mathbf{i}+2\mathbf{j}+\mathbf{k}$ exists over a surface S defined by $x^2+y^2+z^2 = 9$ bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant,

Evaluate $\int_S \mathbf{F}.dS$ over the surface indicated.



$dS = \hat{\mathbf{n}}dS$; $\hat{\mathbf{n}} = \dfrac{\nabla S}{|\nabla S|}$

$S : x^2 + y^2 + z^2 - 9 = 0$

$\nabla S = \dfrac{\partial S}{\partial x}\mathbf{i} + \dfrac{\partial S}{\partial y}\mathbf{j} + \dfrac{\partial S}{\partial z}\mathbf{k} = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$

$\therefore |\nabla S| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{9} = 6$

$\therefore\ \hat{\mathbf{n}} = \dfrac{1}{6}(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) = \dfrac{1}{3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$\int_S \mathbf{F}.dS = \int_S \mathbf{F}.\hat{\mathbf{n}}dS = \int_S (y\mathbf{i}+2\mathbf{j}+\mathbf{k}).\frac{1}{3}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})dS = \frac{1}{3}\int_S (xy + 2y + z)\, dS$

Before integrating over the surface, we convert to spherical polar coordinates.

$x = 3\sin\phi\cos\theta$; $y=3\sin\phi\sin\theta$ $z = 3\cos\phi$; $dS=9\sin\phi d\phi d\theta$

Limits of ϕ and θ are $\phi = 0$ to $\frac{\pi}{2}$; $\theta = 0$ to $\frac{\pi}{2}$

$\int_S \mathbf{F}.dS = \frac{1}{3}\int_S (xy + 2y + z)\, dS$

$xy = 3\sin\phi\cos\theta \,.\, 3\sin\phi\sin\theta = 9\sin^2\theta\cos\theta$

$2y = 2 \,.\, 3\sin\phi\sin\theta = 6\sin\phi\sin\theta$

$z = 3\cos\phi$

$dS = 9\sin\phi\, d\phi\, d\theta$

Putting these values we get

$$\therefore \int_S \mathbf{F}.dS = \frac{1}{3}\int_0^{\pi/2}\int_0^{\pi/2} (9\sin^2\phi\sin\theta\cos\theta+6\sin\phi\sin\theta+3\cos\phi)9\sin\phi d\phi d\theta$$

$$= 9\int_0^{\pi/2}\int_0^{\pi/2} (3\sin^3\phi\sin\theta\cos\theta+2\sin^2\phi\sin\theta+\sin\phi\cos\phi)d\phi d\theta$$

As we know that

$\int_0^{\pi/2} \sin^3\phi\, d\phi = \frac{2}{3}$ by Wallis Formula

also $\int_0^{\pi/2} \sin^2\phi\, d\phi = \frac{1}{2}\frac{\pi}{2}$

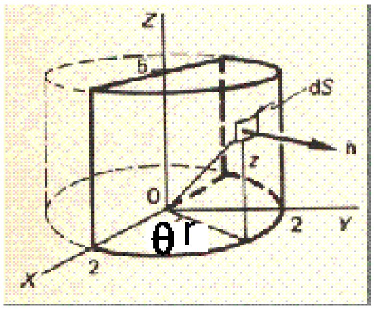
So we get

$$\int_S \mathbf{F}.d\mathbf{S} = 9\int_0^{\pi/2}\left(2\sin\theta\cos\theta + \frac{\pi}{2}\sin\theta + \frac{1}{2}\right)d\theta = 9\left[\sin^2\theta - \frac{\pi}{2}\cos\theta + \frac{\theta}{2}\right]_0^{\pi/2} = 9\left[(1 - 0 + \frac{\pi}{4}) - (0 - \frac{\pi}{2} + 0)\right]$$

$$= 9\left[1 + \frac{\pi}{4} + \frac{\pi}{2}\right] = 9\left(1 + \frac{3\pi}{4}\right)$$

Example

Evaluate $\int_S \mathbf{F}.d\mathbf{S}$ where F = 2y**j** + z**k** and S is the surface $x^2 + y^2 = 4$ in the first two octants bounded by the planes z = 0, z = 5 and y = 0.



$S : x^2 + y^2 - 4 = 0$

$$\hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

$$\nabla S = \frac{\partial \mathbf{S}}{\partial x}\mathbf{i} + \frac{\partial \mathbf{S}}{\partial y}\mathbf{j} + \frac{\partial \mathbf{S}}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{i}$$

$$\therefore |\nabla \mathbf{S}| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

$$\therefore \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{4} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j})$$

$$\therefore \int_S \mathbf{F}.d\mathbf{S} = \int_S \mathbf{F}.\hat{\mathbf{n}}\, dS$$

$$\int_S \mathbf{F}.\hat{\mathbf{n}}\, d\mathbf{S} = \frac{1}{2}\int_S (2y^2)\, dS = \int_S y^2\, dS$$

This is clearly a case for using cylindrical polar coordinates.

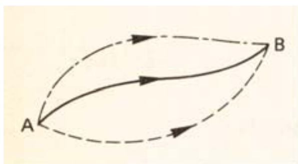
$x = 2\cos\theta$; $y = 2\sin\theta$ $z = z$; $dS = 2d\theta\, dz$

$$\therefore \int_S \mathbf{F}.d\mathbf{S} = \int_S y^2 dS = \iint_S 4\sin^2\theta 2d\theta dz = 8\iint_S \sin^2\theta\, d\theta\, dz$$

Limits: $\theta = 0$ to $\theta = \pi$; $z = 0$ to $z = 5$

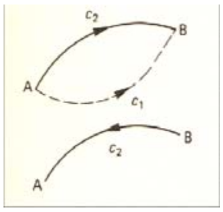
$$\int_S \mathbf{F}.d\mathbf{S} = 4\int_{z=0}^{5}\int_{\theta=0}^{\pi}(1 - \cos 2\theta)\, d\theta dz = 4\int_0^5 \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} dz = 4\int_0^5 \pi\, dz = 4\pi\left[z\right]_0^5 = 20\pi$$

Conservative Vector Fields



In general, the value of the integral $\int_C \mathbf{F}.d\mathbf{r}$ between two stated points A and B depends on the particular path of integration followed. If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field F is said to be conservative.

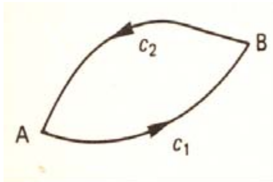
It follows that, for a closed path in a conservative field, $\oint_C \mathbf{F}.d\mathbf{r} = 0$.



For, if the field is conservative,

$\int_{C_1(AB)} \mathbf{F}.d\mathbf{r} = \int_{c_2(AB)} \mathbf{F}.d\mathbf{r}$

But $\int_{c_2(BA)} \mathbf{F}.d\mathbf{r} - \int_{c_2(AB)} \mathbf{F}.d\mathbf{r}$



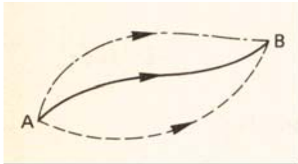
Hence, for the closed path $AB_{c_1} + BA_{c_2}$, $\oint \mathbf{F}.d\mathbf{r}$

$= \int_{C_1(AB)} \mathbf{F}.d\mathbf{r} + \int_{c_2(AB)} \mathbf{F}.d\mathbf{r} = \int_{C_1(AB)} \mathbf{F}.d\mathbf{r} - \int_{c_2(AB)} \mathbf{F}.d\mathbf{r}$

$= \int_{C_1(AB)} \mathbf{F}.d\mathbf{r} - \int_{c_1(AB)} \mathbf{F}.d\mathbf{r} = 0$

Lecture No -38 Vector Field

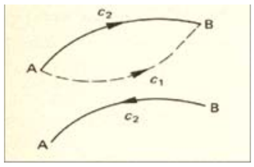
Conservative Vector Fields



In general, the value of the integral $\int_C \mathbf{F.dr}$ between two stated points A and B depends on the particular path of integration followed. If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field F is said to be conservative.

It follows that, for a closed path in a conservative field, $\oint_C \mathbf{F.dr} = 0$.

For, if the field is conservative,

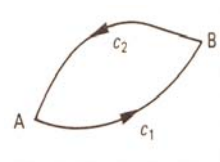


$\int_{C_1(AB)} \mathbf{F.dr} = \int_{c_2(AB)} \mathbf{F.dr}$ But $\int_{c_2(BA)} \mathbf{F.dr} - \int_{c_2(AB)} \mathbf{F.dr}$

Hence, for the closed path $AB_{c_1} + BA_{c_2}$, $\oint \mathbf{F.dr}$

$= \int_{C_1(AB)} \mathbf{F.dr} + \int_{c_2(AB)} \mathbf{F.dr}$

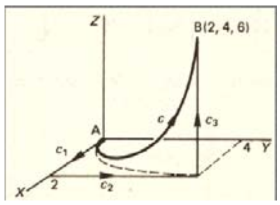
$= \int_{C_1(AB)} \mathbf{F.dr} - \int_{c_2(AB)} \mathbf{F.dr} = \int_{C_1(AB)} \mathbf{F.dr} - \int_{c_1(AB)} \mathbf{F.dr} = 0$



Note that this result holds good only for a closed curve and when the vector field is a conservative field.
Now for some examples

Example

If $F = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$, evaluate the line integral $\int \mathbf{F}.dr$ between A(0,0,0) and B(2,4,6)
(a) along the curve c whose parametric equations are $x = u$, $y = u^2$, $z = 3u$
(b) along the three straight lines c_1:(0,0,0) to (2, 0, 0); c_2 : (2, 0, 0) to (2, 4, 0); c_3: (2, 4, 0) to (2, 4, 6).
Hence determine whether or not F is a conservative field.
First draw the diagram.



(a) $F = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

$x = u$; $y = 2u$; $z = 3u$

$\therefore$ $dx = du$; $dy = 2udu$; $dz = 3du$.

$\mathbf{F.dr}=(2xyz\mathbf{i}+x^2z\mathbf{j}+x^2y\mathbf{k}).(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$

$= 2xyz\,dx + x^2z\,dy + x^2y\,dz$

Using the transformation shown above, we can now express $\mathbf{F}.dr$ in terms of u.

for $2xyzdx=(2u)(u^2)(3u)du = 6u^4du$

$x^2zdy = (u2)(3u)(2u)du = 6u^4du$

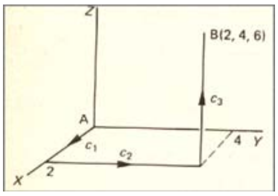
$x^2ydz = (u^2)(u^2)3du = 3u^4du$

$\therefore$ $\mathbf{F}.dr = 15u^4du$

The limits of integration in u are $u = 0$ to $u = 2$

$$\therefore \int_C F.dr = \int_0^2 15u^4du = \left[3u^5\right]_0^2 = 96$$

(b) The diagram for (b) is as shown. We consider each straigh



$\int \mathbf{F}.\mathrm{dr} = \int(2xyz\,dx + x^2zdy+x^2ydz)$

c_1: (0,0,0) to (2,0,0); y = 0, z = 0, dy = 0,

$\therefore \quad \int_{C_1} \mathbf{F}.\mathrm{dr} = 0 + 0 + 0 = 0$

In the same way, we evaluate the line integral along c_2 and c_3.

$\int_{C_1} \mathbf{F}.\mathrm{dr} = 0;$

$\int \mathbf{F}.\mathrm{dr}=\int(2xyzdx+x^2zdy+x^2ydz)$

c_2: (2,0,0) to (2,4,0); x = 2, z = 0, dx=0, dz = 0

$\therefore \quad \int_{C_2} \mathbf{F}.\mathrm{dr} = 0+0+0 = 0 \int_{C_2} \mathbf{F}.\mathrm{dr} = 0$

c_3: (2,4,0) to (2,4,6); x = 2, y = 4, dx=0, dy = 0

$\therefore \quad \int_{C_3} \mathbf{F}.\mathrm{dr} = 0+0+\int_0^6 16dz=\left[16z\right]_0^6 = 96 \quad \therefore \int_{C_3} \mathbf{F}.\mathbf{dr} = 96$

Collecting the three results together

$\int_{c_1+c_2+c_3} \mathbf{F}.\mathbf{dr} = 0 + 0 + 96 \quad \therefore \int_{c_1+c_2+c_3} \mathbf{F}.\mathbf{dr} = 96$

In this particular example, the value of the line integral is independent of the two paths we have used joining the same two end points and indicates that F is a conservative field. It follows that

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix}$$

$= (x^2-x^2)\mathbf{i}-(2xy-2xy)\mathbf{j}+(2xz-2xz)\mathbf{k} = 0$

$\therefore$ curl $\mathbf{F} = 0$

So three tests can be applied to determine whether or not a vector field is conservative. They are

(a) $\oint \mathbf{F}.\mathbf{dr} = 0$

(b) curl $\mathbf{F} = 0$

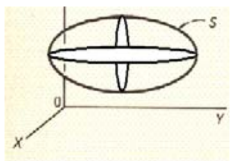
(c) $\mathbf{F}$ = grad V

Any one of these conditions can be applied as is convenient.

Divergence Theorem (Gauss' theorem)

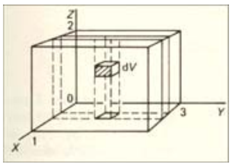
For a closed surface S, enclosing a region V in a vector field F

$\int_V \text{div } \mathbf{F}\, dV = \int_S \mathbf{F}.\mathbf{dS}$



In general, this means that the volume integral (tripple integral) on the left-hand side can be expressed as a surface integral (double integral) on the right-hand side.

Example



Verify the divergence theorem for the vector field
$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ taken over the region bonded by the planes $z = 0, z = 2, x = 0, x = 1, y = 0, y = 3$.
$dV = dx\,dy\,dz$

we have to show that $\int_V \text{div}\,\mathbf{F}\,dV = \int_S \mathbf{F.dS}$

(a) To find $\int_V \text{div}\,\mathbf{F}\,dV$

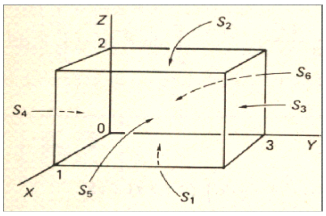
$$\text{div F} = \nabla.\text{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\mathbf{.}\,(x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y) = 2x+0+0 = 2x$$

$$\therefore \int_V \text{div F}\,dV = \int_V 2x\,dv = \iiint_V 2x\,dz\,dy\,dx$$

$$\int_V \text{div}\,\mathbf{F}\,dV = \int_0^1\int_0^3\int_0^2 2x\,dz\,dy\,dx = \int_0^1\int_0^3 \Big[2xz\Big]_0^2 dy\,dx = \int_0^1 \Big[2xz\Big]_0^3 dx = \int_0^1 12x\,dx = \Big[6x^2\Big]_0^1 = 6$$

(b) to find $\int_S \text{F.dS}$ i.e. $\int_S \mathbf{F}.\hat{\mathbf{n}}\,dS$



The enclosing surface S consists of six separate plane faces denoted as
$S_1, S_2, \ldots.. S_6$ as shown. We consider each face in turn.
$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(i) S_1 (base): $z = 0$; $\hat{\mathbf{n}} = -\mathbf{k}$ (outwards and downwards)

$\therefore \quad \mathbf{F} = x^2\mathbf{i} + y\mathbf{k} \quad dS_1 = dx\,dy$

$$\therefore \int_{S_1} \mathbf{F}.\,\hat{\mathbf{n}}dS = \iint_{S_1} (x2\mathbf{i}+y\mathbf{k}).(-\mathbf{k})dydx = \int_0^1\int_0^3 (-y)\,dydx = \int_0^1 \left[-\frac{y^2}{2}\right]_0^3 dx = -\frac{9}{2}$$

(ii) S_2 (top): $z = 2$: $\hat{\mathbf{n}} = \mathbf{k}$ $dS_2 = dx\,dy$

$$\int_{S_2} \mathbf{F}.\hat{\mathbf{n}}dS = \iint_{S_2} (x^2\mathbf{i}+z\mathbf{j}+y\mathbf{k}).(\mathbf{k})dydx = \int_0^1\int_0^3 y\,dy\,dx = \frac{9}{2}$$

(iii) S_3 (right-hand end): $y = 3$;

$\hat{\mathbf{n}} = \mathbf{j}$ $dS_3 = dxdz\mathbf{j}$

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

$$\therefore \int_{S_3} \mathbf{F}.\hat{\mathbf{n}}dS = \iint_{S_3} (x^2\mathbf{i}+z\mathbf{j}+3\mathbf{k}).(\mathbf{j})dzdx = \int_0^1\int_0^2 z\,dz\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^2 dx = \int_0^1 2\,dx = 2$$

(iv) S_4 (left-hand end): $y = 0$, $\hat{\mathbf{n}} = -\mathbf{j}$, $dS_4 = dx\,dz$

$$\therefore \int_{S_4} \mathbf{F}.\hat{\mathbf{n}}dS = -2$$

$$\text{for} \int_{S_4} \mathbf{F}.\hat{\mathbf{n}}dS = \iint_{S_4} (x^2\mathbf{i}+z\mathbf{j}+y\mathbf{k}).(-\mathbf{j})dzdx = \int_0^1\int_0^2 (-z)\,dz\,dx = \int_0^1 \left[-\frac{z^2}{2}\right]_0^2 dx$$

$$= \int_0^1 (-2)\,dx = -2$$

Now for the remaining two sides S_5 and S_6. Evaluate these in the same manner, obtaining

$$\int_{S_5} \mathbf{F}.\hat{\mathbf{n}}\,dS = 6; \qquad \int_{S_5} \mathbf{F}.\hat{\mathbf{n}}\,dS = 0$$

Check:

(v) S_5 (front): x = 1; $\hat{\mathbf{n}} = \mathbf{i}$ dS_5 = dy dz

$\therefore \int_{S_5} F.\,\hat{\mathbf{n}}dS = \iint_{S_5} (\mathbf{i}+z\mathbf{j}+y\mathbf{k}).(\mathbf{i})dy\,dz = \iint_{S_5} 1\, dy\, dz = 6$

(vi) S_6 (back): x = 0; $\hat{\mathbf{n}} = -\mathbf{i}$ dS_6=dy dz

$\therefore \int_{S_6} F.\,\hat{\mathbf{n}}dS = \iint_{S_6} (z\mathbf{j}+y\mathbf{k}).(-\mathbf{i})dy\,dz = \iint_{S_6} 0\, dy\, dz = 0$

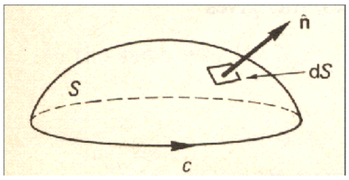
For the whole surface S we therefore have $\int_S \mathbf{F}.dS = -\frac{9}{2}+\frac{9}{2}+2-2+6+0 = 6$

and from our previous work in section (a) $\int_V \text{div}\,\mathbf{F}\, dV = 6$

We have therefore verified as required that, in this example $\int_V \text{div}\,\mathbf{F}\, dV = \int_S \mathbf{F.dS}$

Stokes Theorem

If F is a vector field existing over an open surface S and around its boundarv closed curve C, then $\int_S \text{curl}\,\mathbf{F}.dS = \oint_C \mathbf{F.dr}$

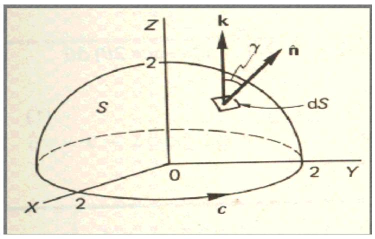


This means that we can express a surface integral in terms of a line integral round the boundary curve. The proof of this theorem is rather lengthy and is to be found in the Appendix. Let us demonstrate its application in the following examples.

Example

A hemisphere S is defined by $x^2+y^2+z^2=4$ ($z \geq 0$). A vector field $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j}+xz\mathbf{k}$ exists over the surface and around its boundary c. Verify Stoke's theorem that

$\int_S \text{curl}\,\mathbf{F}.dS = \oint_C \mathbf{F.dr}$.



S : $x^2 + y^2 + z^2 - 4 = 0$

$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$ c is the circle $x^2 + y^2 = 4$.

(a) $\oint_C \mathbf{F.dr} = \int_C (2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k})\,.\,(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz)$

$= \int_C (2ydx - x\,dy+xz\,dz)$

Converting to polar coordintes.x = 2 cos θ; y = 2 sin θ; z = 0

dx = – 2 sin θ dθ; dy = 2 cos θ dθ; Limits θ = 0 to 2π

$$\oint_C \mathbf{F.dr} = \int_0^{2/\pi} (4\sin\theta[(-2\sin\theta d\theta]-2\cos\theta 2\cos\theta d\theta = -4\int_0^{2/\pi} (2\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= -4\int_0^{2/\pi} (1 + \sin^2\theta)\, d\theta = -2\int_0^{2/\pi} (3 - \cos 2\theta)\, d\theta = -2\left[3\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -12\,\pi \quad (1)$$

(b) Now we determine $\int_S \text{curl}\,\mathbf{F.dS}$

$\int \text{curl}\,\mathbf{F.dS} = \int \text{curl}\,\mathbf{F}.\,\hat{\mathbf{n}}dS$

$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -x & xz \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}\,(z-0) + \mathbf{k}\,(-1-2) \;= -\,z\mathbf{j} - 3\mathbf{k}$$

Now $\hat{\mathbf{n}} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{2}$

Then $\int_S \text{curl } \mathbf{F}.\,\hat{\mathbf{n}}dS = \int_S (-z\mathbf{k}-3\mathbf{k}).\left(\dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2}\right) dS \quad = \dfrac{1}{2}\int_S (-\,yz - 3z)\, dS$

$x = 2 \sin\phi \cos\theta$; $y = 2 \sin\phi \sin\theta$; $z = 2\cos\phi$,$dS = 4 \sin\phi \, d\phi \, d\theta$

$\therefore \int S \text{ curl}\mathbf{F}.\hat{\mathbf{n}}dS = \dfrac{1}{2}\iint_S (-2 \sin\phi \sin\theta\, 2\cos\phi - 6 \cos\phi)\, 4 \sin\phi \, d\phi \, d\theta$

$$= -\int_0^{2\pi}\int_0^{\pi/2} (2\sin^2\phi \cos\phi \sin\theta + 3\sin\phi\cos\phi)\,d\phi\, d\theta = -4\int_0^{2\pi}\left[\frac{2\sin^3\phi \sin\theta}{3} + \frac{3\sin^2\phi}{2}\right]_0^{\pi/2} d\theta$$

$$= -\,4\int_0^{2\pi}\left(\frac{2}{3}\sin\theta + \frac{3}{2}\right) d\theta = -\,12\pi \quad (2)$$

So we have from our two results (1) and (2)

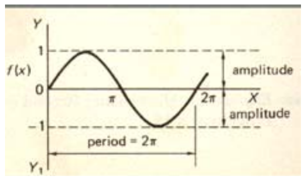
$$\int_S \text{curl } \mathbf{F}.d\mathbf{S} = \oint_C \mathbf{F}.d\mathbf{r}$$

Lecture No -39 Periodic Functions

Periodic functions

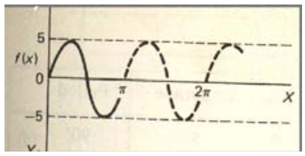
A function f(x) is said to be periodic if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the period of the oscillations. $f(x + p) = f(x)$

Graphs of y = A sin nx



(a) y = sin x The obvious example of a periodic function is $y = \sin x$, which goes through its complete range of values while x increases from 0° to 360°.

The period is therefore 360° or 2π radians and the amplitude, the maximum displacement from the position of rest.



$y = 5\sin 2x$

The amplitude is 5. The period is 180° and there are thus 2 complete cycles in 360°.

Example

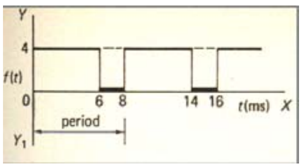
Functions	Amplitude	Period
$y = 3 \sin 5x$	3	72°
$y = 2 \cos 3x$	2	120°
$y = \sin \frac{x}{2}$	1	720°
$y = 4 \sin 2x$	4	180°

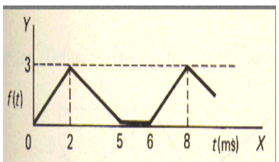
$y = A \sin nx$

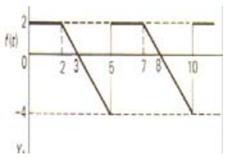
Thanking along the same lines, the function $y = A \sin nx$ has amplitude = A; period $= \frac{360°}{n} = \frac{2\pi}{n}$; n cycles in 360°.

Graphs of $y = A \cos nx$ have the same characteristics

Example





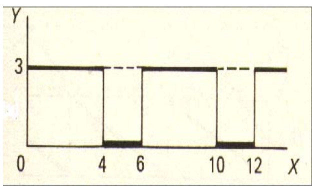


period = 8 ms period = 6 ms period = 5 cm

Analytical description of a periodic function

A periodic function can be defined analytically in many cases.

Example



(a) Between x = 0 and x = 4, y = 3,
i.e. $f(x) = 3 \quad 0 < x < 4$

(b) Between x = 4 and x = 6, y = 0,
i.e. $f(x) = 0 \quad 4 < x < 6.$

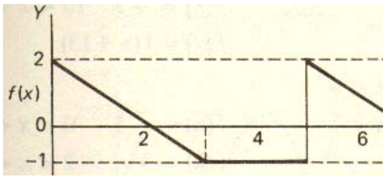
So we could define the function by

$f(x) = 3 \qquad 0 < x < 4$

$f(x) = 0 \qquad 4 < x < 6$

$f(x) = f(x + 6)$

the last line indicating that the function is periodic with period 6 units



$f(x) = 2 - x \qquad 0 < x < 3$

$f(x) = -1 \qquad 3 < x < 5$

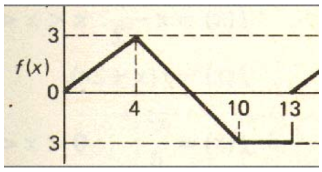
$f(x) = f(x+5)$

$f(x) = \dfrac{3x}{4} \qquad 0 < x < 4$

$f(x) = 7 - x \qquad 4 < x < 10$

$f(x) = -3 \qquad 10 < x < 13$

$f(x) = f(x + 13)$



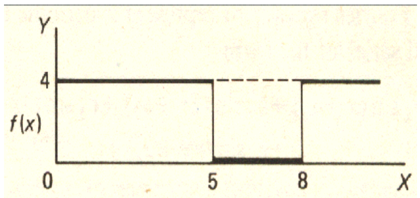
Example

Sketch the graphs of the following inserting relevant values.

1. $f(x) = 4 \qquad 0 < x < 5$

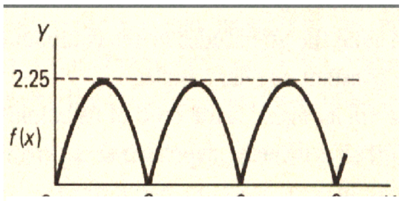
   $f(x) = 0 \qquad 5 < x < 8$

   $f(x) = f(x + 8)$



   $f(x) = 3x - x^2 \qquad 0 < x < 3$

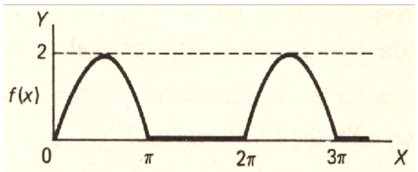
   $f(x) = f(x + 3)$



   $f(x) = 2\sin x \qquad 0 < x < \pi$

   $f(x) = 0 \qquad \pi < x < 2\pi$

   $f(x) = f(x + 2\pi)$.

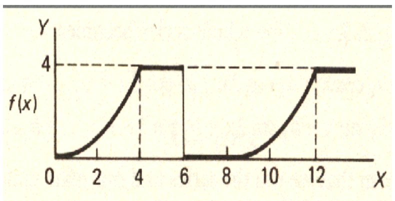


   $f(x) = \dfrac{x^2}{4} \qquad 0 < x < 4$

   $f(x) = 4 \qquad 4 < x < 6$

   $f(x) = 0 \qquad 6 < x < 8$

   $f(x) = f(x + 8)$.



Useful integrals

The following integrals appear frequently in our work on Fourier series, so it will help if we obtain the result in readiness. In each case, m and n are integers other than zero.

(a) $\displaystyle\int_{-\pi}^{\pi} \sin nx\, dx = \left[\frac{-\cos nx}{n}\right]_{-\pi}^{\pi} = \frac{1}{n}\{-\cos n\pi + \cos n\pi\} = 0$

(b) $\displaystyle\int_{-\pi}^{\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = \frac{1}{n}\{\sin n\pi + \sin n\pi\} = 0$

(c) $\displaystyle\int_{-\pi}^{\pi} \sin^2 nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2nx)\, dx = \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi \ (n \neq 0)$

(d) $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx)\, dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi \quad (n \neq 0)$

(e) $\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\sin(n+m)x + \sin(n-m)x\} dx = \frac{1}{2}\{0 + 0\} = 0$

from result (a) with $n \neq m$

(f) $\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\cos(n+m)x + \cos(n-m)x\} dx = \frac{1}{2}\{0 + 0\} = 0$

from result (b) with $n \neq m$

Note:
If n = m, then

$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx$ becomes $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi$ from (d) above.

(g) $\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = -\frac{1}{2}\int_{-\pi}^{\pi} (-2) \sin nx \sin mx dx = -\frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x - \cos(n-m)x dx$

$= \frac{1}{2}\int_{-\pi}^{\pi} \{0 - 0) = 0$ from result (b) with $n \neq m$

Note:
If n = m, then

$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx$ becomes $\int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$ from (c) above

Summary of integrals

(a) $\int_{-\pi}^{\pi} \sin nx\, dx = 0$, (b) $\int_{-\pi}^{\pi} \cos nx\, dx = 0$, (c) $\int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi\ (\neq 0)$

(d) $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi\ (n \neq 0)$ (e) $\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0$

(f) $\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = 0 \quad (n \neq m)$ (g) $\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 0 \quad (n \neq m)$

$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \pi \quad (n = m)$ $\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \pi \quad (n = m)$

Note
We have evaluated the integrals between $-\pi$ and π, but, provided integration is carried out over a complete periodic interval of 2π, the results are the same. Thus, the limits could just as well be $-\pi$ to π, 0 to 2π, $-\pi/2$ to $3\pi/2$, etc. We can therefore choose the limits to suit the particular problems.

Fourier series

Periodic functions of period 2π

The basis of a Fourier series is to represent a periodic function by a trigonometrical series of the form.

$f(x) = A_0+c_1 \sin(x+\alpha_1) + c_2 \sin(2x+\alpha_2) + c_3 \sin(3x+\alpha_3)+\ldots+c_n \sin(nx+\alpha_n)+ \ldots$

where A_0 is a constant term

$c_1, c_2, c_3 \ldots c_n$ denote the amplitudes of the compound sine terms $\alpha_1, \alpha_2, \alpha_3 \ldots$ are constant auxiliary angles.

Each sine term, $c_n \sin(nx + \alpha_n)$ can be expanded thus:

$c_n \sin(nx + \alpha_n) = c_n\{\sin nx \cos\alpha_n+\cos n x \sin\alpha_n\}= (c_n\sin\alpha_n) \cos nx + (c_n\cos\alpha_n) \sin nx$

$= a_n\cos nx + b_n\sin nx$

the whole series becomes.

$$\mathbf{f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}}$$

Lecture No- 40 Fourier Series

As we know that $A_0 + \sum_{n=1}^{\infty} \{a_n\cos nx + b_n\sin nx\}$;

which can be written as in the expanded from

$A_0+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x)+ .... + (a_n\cos nx+b_n\sin nx)+..........$

$A_0+a_1\cos x+a_2\cos 2x+.+a_n\cos nx+.. .. +b_1\sin x+b_2\sin 2x+... b_n\sin nx+..........$

$f(x) = A_0+a_1\cos x+a_2\cos 2x+.+a_n\cos nx+.. ....+b_1\sin x+b_2\sin 2x+... b_n\sin nx+...$

Fourier coefficients

We have defined Fourier series in the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n\cos nx + b_n\sin nx\}; \qquad \text{n a positive integer}$$

(a) To find A_0, we integrate f(x) with respect to x from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} A_0\, dx + \sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nx\, dx+\int_{-\pi}^{\pi} b_n \sin nx\, dx\right\} = [A_0x]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} \{ 0 + 0 \} = 2A_0\pi$$

$$2A_0\,\pi = \int_{-\pi}^{\pi} f(x)\, dx$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 1/2\ a_0\ ; \qquad \text{Where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

(b) To find a_n we multiply f(x) by cos mx and integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = \int_{-\pi}^{\pi} A_0 \cos mx\, dx+\sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nx \cos mx\, dx+\int_{-\pi}^{\pi} b_n \sin nx \cos mx\, dx\right\}$$

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = A_0\,\{0\} + \sum_{n=1}^{\infty}\{a_n\,(0) + b_n\,(0)\}= 0 \qquad \text{for } n \neq m$$

$= 0 + a_n\pi + 0 = a_n\pi$ for n = m

$$\therefore\ \mathbf{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx}$$

(c) To find b_n we multiply f(x) by sinmx and integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\sin mx\, dx = \int_{-\pi}^{\pi} A_0 \sin mx\, dx +\sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nx\sin mx\, dx+\int_{-\pi}^{\pi} b_n \sin nx \sin mx dx\right\}$$

$$\int_{-\pi}^{\pi} f(x)\sin mx\, dx = A_0\,\{0\} + \sum_{n=1}^{\infty}\{a_n\,(0) + b_n\,(0)\}= 0 \qquad \text{for } n \neq m$$

$= 0 + 0 + b_n\pi = b_n\pi$ for n = m

$$\therefore\ \mathbf{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx}$$

Result For Fourier Series

$$\mathbf{f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n\cos nx+b_n\sin nx\};}$$

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ = 2×mean value of f(x) over a period

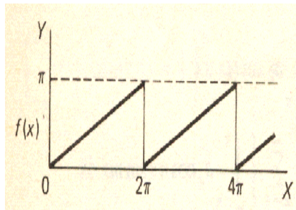
(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$=2×mean value of f(x) cosnx over a period.

(c) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ = 2 × mean value of f(x) sin nx over a period.

In each case, n = 1, 2, 3, ……

Example

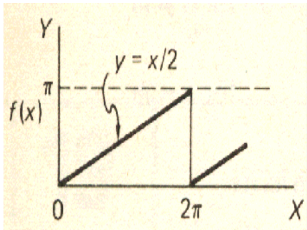
Determine the Fourier series to represent the periodic function shown.



It is more convenient here to take the limits as 0 to 2π.
The function can be defined as

$f(x) = \frac{x}{2}$ $\quad 0 < x < 2\pi$

$f(x) = f(x + 2\pi)$ $\quad$ period = 2π.



Now to find the coefficients

(a) $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)dx = \frac{1}{4\pi}\left[x^2\right]_0^{2\pi} = \pi$

$a_0 = \pi$

(b) $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\cos nx\,dx$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{2\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right\} = \frac{1}{2\pi}\left\{(0-0) - \frac{1}{n}(0)\right\}$$

$\therefore a_n = 0$

(a) $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$ $\quad$ So we now have

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}\sin nx\,dx = \frac{1}{2\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\cos nx\,dx\right\} = -\frac{1}{2\pi n}[2\pi - 0] = -\frac{1}{n}$$

$$\mathbf{a_0 = \pi; \qquad a_n = 0; \qquad b_n = -\frac{1}{n}}$$

Now the general expression for a Fourier series is

$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\}$ Therefore in this case

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \{b_n \sin nx\} = \frac{\pi}{2} + \{-\frac{1}{1}\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x - ...\} \quad \text{since } a_n = 0$$

$$f(x) = \frac{\pi}{2} - \{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...\}$$

Dirichlet Conditions

If the Fourier series is to represent a function f(x), then putting $x = x_1$ will give an infinite series in x_1 and the value of this should converge to the value of $f(x_1)$ as more and more terms of the series are evaluated. For this to happen, the following conditions must be fulfilled.

(a) The function *f*(x) must be defined and single-valued.

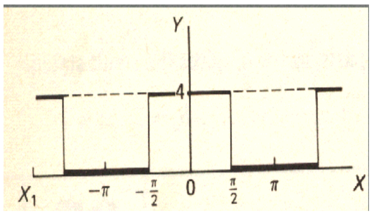
(b) *f*(x) must be continuous or have a finite number of finite discontinuities within a periodic interval.

(c) *f*(x) and $f'(x)$ must be piecewise continuous in the periodic interval.

If these Dirichlet conditions are satisfied, the Fourier series converges to $f(x_1)$, if $x = x_1$ is a point of continuity

Example

Find the Fourier series for the function shown.



Consider one cycle between x=0 and x=π.

The function can be defined by

$$f(x) = 0 \qquad -\pi < x < -\frac{\pi}{2}$$

$$f(x) = 4 \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(x) = 0 \qquad \frac{\pi}{2} < x < \pi$$

$$f(x) = f(x + 2\pi)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

The expression for a_0 is $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ This gives

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} 0\,dx + \int_{-\pi/2}^{\pi/2} 4dx + \int_{\pi/2}^{\pi} 0\,dx\right\} = \frac{1}{\pi}[4x]_{-\pi/2}^{\pi/2} = 4$$

$$a_0 = 4$$

(b) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2}(0)\cos nx dx + \int_{-\pi/2}^{\pi/2} 4\cos nx dx + \int_{\pi/2}^{\pi}(0)\cos nx dx\right\} = \frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2} 4\cos nx dx\right\}$$

$$= \frac{4}{\pi n}\sin nx\Big|_{-\pi/2}^{\pi/2} = \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

$$a_n = \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

Then considering different integer values of n, we have

If n is even $\qquad a_n = 0$

If n = 1.5, 9…. $\qquad a_n = \frac{8}{n\pi}$

If n = 3, 7, 11, …. $a_n = -\frac{8}{n\pi}$

(c) To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2}(0)\sin nx dx + \int_{-\pi/2}^{\pi/2} 4\sin nx dx + \int_{\pi/2}^{\pi}(0)\sin nx dx\right\}$$

$$b_n = \frac{4}{\pi}\int_{-\pi/2}^{\pi/2}\sin nx\,dx = \frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi/2}^{\pi/2} = -\frac{4}{n\pi}\left\{\cos\frac{n\pi}{2} - \cos\left(\frac{-n\pi}{2}\right)\right\} = 0$$

$b_n = 0$

So with $a_0 = 4$: an as stated above; $b_n = 0$;

The Fourier series is

$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots..\right\}$$

In this particular example, there are, in fact, no sine terms.

Effect Of Harmonics

It is interesting to see just how accurately the Fourier series represents the function with which it is associated. The complete representation requires an infinite number of terms, but we can, at least, see the effect of including the first few terms of the series.

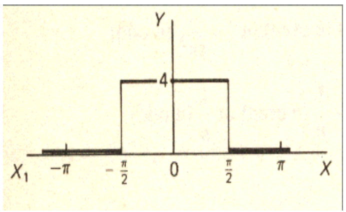
Let us consider the waveform shown. We established earlier that the function

$f(x) = 0 \qquad -\pi < x < -\frac{\pi}{2}$

$f(x) = 4 \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$f(x) = 0 \qquad \frac{\pi}{2} < x < \pi$

$f(x) = f(x + 2\pi)$

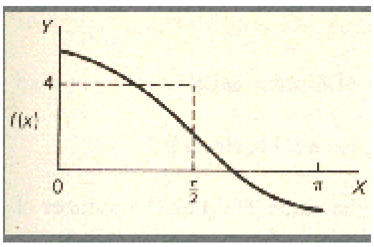


gives the Fourier series

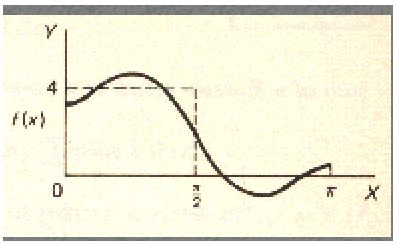
$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{3}\cos 5x - \frac{1}{7}\cos 7x + \ldots.\right\}$$

If we start with just one cosine term, we can then see the effect o including subsequent harmonics. Let us restrict our attention to just the right-hand half of the symmetrical waveform. Detailed plotting of points gives the following development.

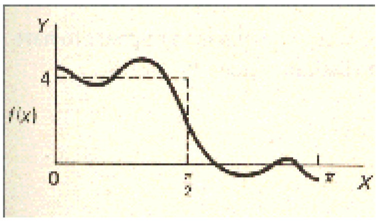
(1) $f(x) = 2 + \frac{8}{\pi}\cos x$



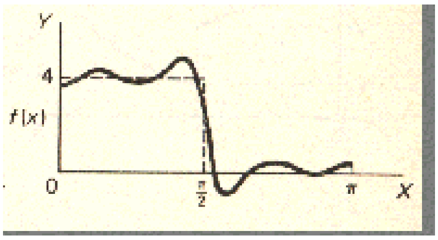
(2) $f(x) = 2 + \frac{8}{\pi}\ \{\cos x - \frac{1}{3}\cos 3x\}$



(3) $f(x)=2+\frac{8}{\pi}\ \{\cos x-\frac{1}{3}\cos 3x + \frac{1}{3}\cos 5x)$



(4) $f(x) = 2 + \frac{8}{\pi}\ \{\cos x-\frac{1}{3}\cos 3x+\frac{1}{3}\cos 5x - \frac{1}{7}\cos 7x\}$



As the number of terms is increased, the graph gradually approaches the shape of the original square waveform. The ripples increase in number and decrease in amplitude, but a perfectly square waveform is unattainable in practice. For practical purpose, the first few terms normally suffice to give an accuracy of acceptable level.

Lecture No -41 Examples

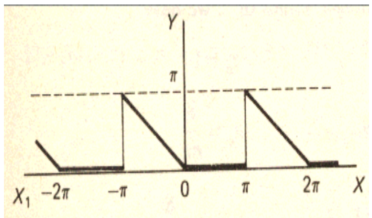
Example

Find the Fourier series for the function defined by

$f(x) = -x \qquad -\pi < x < 0$

$f(x) = 0 \qquad 0 < x < \pi$

$f(x) = f(x + 2\pi)$



The general expressions for a_0, a_n, b_n are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\,dx = \frac{1}{\pi}\left[-\frac{x^2}{2}\right]_{-\pi}^{0} = \frac{\pi}{2}$$

(b) To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx = -\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx\right\} = -\frac{1}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{1}{n}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{1}{\pi n^2}\left\{\left[\frac{\cos nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{1}{\pi n^2}\left[\cos 0 - \cos n\pi\right]$$

$$= -\frac{1}{\pi n^2}\{1 - \cos n\pi\}$$

But $\cos n\pi = 1$ (n even) and $\cos n\pi = -1$ (n odd)

$a_n = -\dfrac{2}{\pi n^2}$ (n odd) and $a_n = 0$ (n even)

(c) Now to find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nx\,dx = -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$= -\frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\right\} = -\frac{1}{\pi}\left\{\frac{\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}\right\} = \frac{\cos n\pi}{n}$$

$b_n = \dfrac{1}{n}$ (n even) and $b_n = -\dfrac{1}{n}$ (n odd)

So we have

$$\mathbf{a_0 = \frac{\pi}{2};\ \ a_n = 0\ (n\ even)\ \ and\ a_n = -\frac{2}{\pi n^2}\quad (n\ odd)}$$

$$\mathbf{b_n = -\frac{1}{n}\ (n\ even)\ \ and\ b_n = \frac{1}{n}\quad (n\ odd)}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right) + \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right)$$

It is just a case of substituting n = 1, 2, 3, etc.

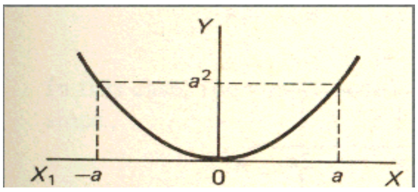
In this particular example, we have a constant term and both sine and cosine terms.

Odd And Even Functions

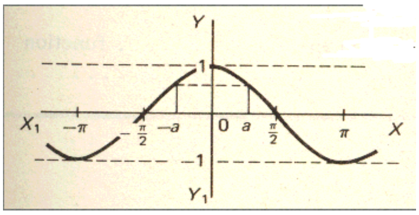
(a) Even functions

A function f(x) is said to be even if $f(-x) = f(x)$ i.e. the function value for a particular negative value of x is the same as that for the corresponding positive value of x. The graph of an even function is therefore symmetrical about the y-axis.

$y = f(x) = x^2$ is an even function since
$f(-2) = 4 = f(2)$
$f(-3) = 9 = f(3)$ etc.

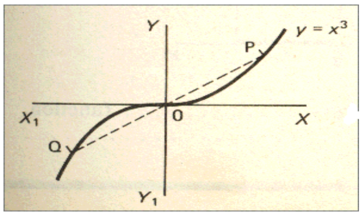


$y = f(x) = \cos x$ is an even function
since $\cos(-x) = \cos x$
$f(-a) = \cos a = f(a)$



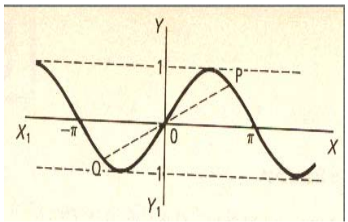
(b) Odd functions

A function f(x) is said to be odd if $f(-x) = -f(x)$
i.e. the function value for a particular negative value of xis numerically equal to that for the corresponding positive value of x but opposite in sign. The graph of an odd function is thus symmetrical about the origin.

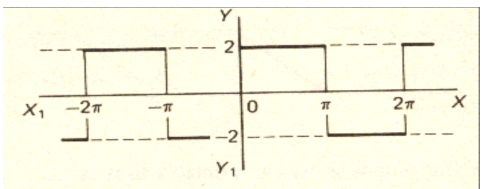


$y = f(x) = x^3$ is an odd function since
$f(-2) = -8 = -f(2)$
$f(-5) = -125 = -f(5)$ etc.

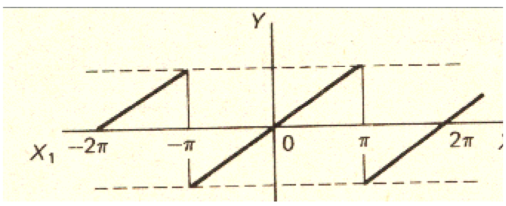
$y = f(x) = \sin x$ is an odd function
Since $\sin(-x) = -\sin x$
$f(-a) = -f(a)$.



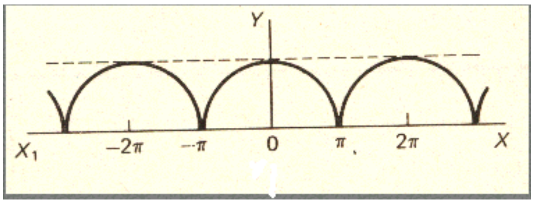
So, for an even function $f(-x) = f(x)$ symmetrical about the y-axis
for an odd function $f(-x) = -f(x)$ symmetrical about the origin.



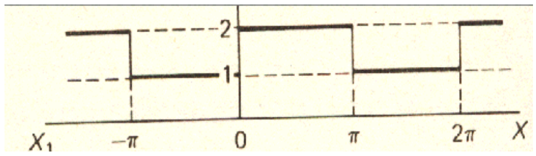
odd.



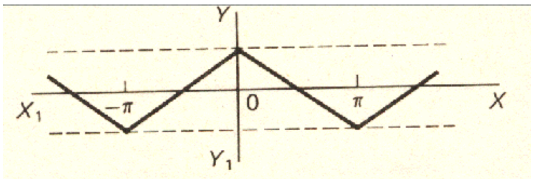
Odd



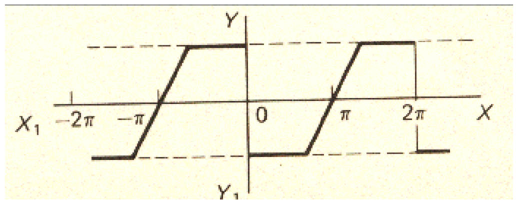
Even



neither



Even



Odd

<u>Products Of Odd And Even Functions</u>

The rules closely resemble the elementary rules of signs.

(even)×(even)=(even) like (+)×(+)=(+) ; (odd)× (odd) = (even) (–)×(–)=(+) ;
(odd)× (even) = (odd) (–)×(+)=(–)

The results can easily be proved.

(a) Two even functions

Let F(x) = f(x) g(x) where f(x) and g(x) are even functions.

Then F(– x) = f(– x) g (– x) = f(x) g(x) since f(x) and g(x) are even.

$\therefore$ F(– x) = F(x)

F(x) is even

(b) Two odd functions

Let F(x) = f(x) g(x) where f(x) and g(x) are odd functions.

Then F(– x) = f(– x) g(– x) = {– f (x)} {– g(x)}

since f(x) and g(x) are odd.

= f(x) g(x) = F (x)

$\therefore$ F(– x) = F(x)

F (x) is even

Finally

(c) One odd and one even function

Let F(x) = f(x) g(x) where f(x) is odd and g(x) even.

Then F (– x) = f (– x) g(– x) = – f(x) g(x) = – F (x)

$\therefore$ F (– x) = – F (x)

F (x) is odd

So if f(x) and g(x) are both even, then f(x) g(x) is even and if f(x) and g(x) are both odd, then f(x) g(x) is even but if either f(x) or g(x) is even and the other odd. Then f(x) g(x) is odd.

Example

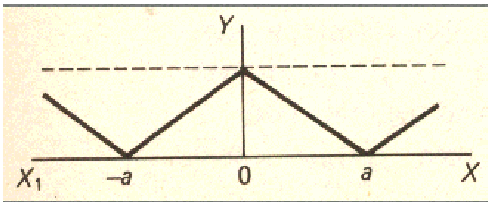
State whether each of the following products is odd, even, or neither.

1. $x^2 \sin 2x$ odd (E) (O) = (O)
2. $x^3 \cos x$ odd (O) (E) = (O)
3. $\cos 2x \cos 3x$ even (E) (E) = (E)
4. $x \sin nx$ even (O) (O) = (E)
5. $3 \sin x \cos 4x$ odd (O) (E) = (O)
6. $(2x + 3) \sin 4x$ neither (N) (O) = (N)
7. $\sin^2 x \cos 3x$ even (E) (E) = (E)
8. $x^3 e^x$ neither (O) (N) = (N)
9. $(x^4 + 4) \sin 2x$ odd (E) (O) = (O)

<u>Two useful facts emerge from odd and even functions.</u>

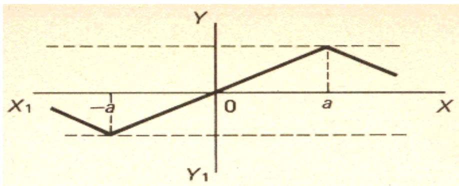
(a) For an even function

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\,dx$$



(b) For an odd function

$$\int_{-a}^{a} f(x)\,dx = 0$$



<u>THEOREM 1</u>

If f(x) is defined over the interval $-\pi < x < \pi$ and f(x) is even, then the Fourier series for f(x) contains cosine terms only. Included in this is a_0 which maybe regarded as $a_n \cos nx$ with n = 0.

Proof:

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx = \dfrac{2}{\pi}\int_{0}^{\pi} f(x)dx \quad \therefore \quad a_0 = \dfrac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$

(b) $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$

But f(x) cos nx is the product of two even functions and therefore itself even.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx. \quad \therefore a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Since f(x) sin nx is the product of an even function and an odd function, it is itself odd.

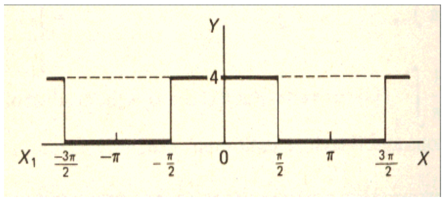
$$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

$\therefore\ b_n = 0$

Therefore, there are no sine terms in the Fourier series for f(x).

Example

The waveform shown is symmetrical about the y-axis. The function is therefore even and there will be no sine terms in the series.



$$\therefore \quad f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{(a)}\ \ a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx. = \frac{2}{\pi}\int_{0}^{\pi/2} 4\,dx$$

$$= \frac{2}{\pi}\left[4x\right]_{0}^{\pi/2} = 4$$

$$\text{(b)}\ \ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi/2} 4\cos nx\,dx = \frac{8}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi/2}$$

$$= \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

But $\sin\frac{n\pi}{2} = 0$ for n even

$= 1$ for n 1, 5, 9, …..

$= -1$ for n = 3, 7, 11, ….

$a_n = 0$ (n even);

$a_n = \frac{8}{\pi n}$ (n = 1, 5, 9, ….);

$a_n = -\frac{8}{\pi n}$ (n = 3, 7, 11 ……)

(c) $b_n = 0$, since f(x) is an even function.Therefore, the required series is

$$f(x) = 2 + \frac{\pi}{8}\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{3}\cos 5x - \frac{1}{7}\cos 7x + \ldots\}$$

Theorem 2:

If f(x) is an odd function defined over the interval $-\pi < x < \pi$, then the Fourier series for f(x) contains sine terms only.

Proof:

Since f(x) is an odd function

$$\int_{-\pi}^{0} \mathbf{f(x)\,dx} = -\int_{0}^{\pi} \mathbf{f(x)\,dx.}$$

$$\text{(a)}\ \ a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

But f(x) is odd

$\therefore \quad a_0 = 0$

$$\text{(b)}\ \ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

Remembering that f(x) is odd and cosnx is even, the product f(x) cosnx is odd.

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\text{odd function})\,dx$$

$\therefore \quad a_n = 0$

(b) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$

and since f(x) and sin nx are each odd, the product f(x) sin nx is even.

Then $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\text{even function})\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$

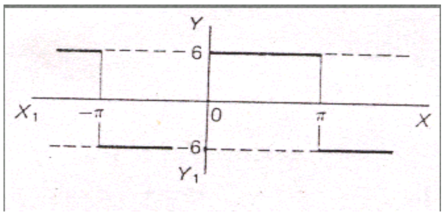
$\therefore \; b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$

So,
If f(x) is odd function then $a_0 = 0$; $a_n = 0$;

$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$ i.e. the Fourier series contains sine terms only.

Example

$f(x) = -6 \qquad -\pi < x < 0$
$f(x) = 6 \qquad 0 < x < \pi$
$f(x) = f(x + 2\pi)$



We can see that this is an odd function;
$a_0 = 0$ and $a_n = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$

f(x) sin nx is a product of two odd functions and is therefore even.

$\therefore \; b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} 6 \sin nx \, dx = \frac{12}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{12}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi}^{0} = \frac{12}{\pi}[\cos 0 - \cos n\pi]$$

$$= \frac{12}{\pi n}(1 - \cos n\pi).$$

$b_n = 0 \qquad (n \text{ even}) \qquad b_n = \frac{24}{\pi n} \qquad (n \text{ odd})$

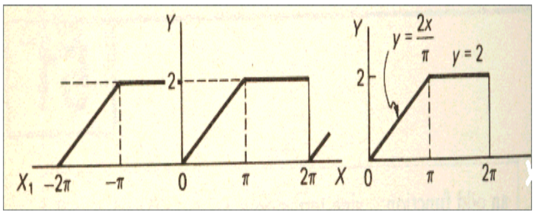
So the series is $f(x) = \frac{24}{\pi}\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\}$

Of course,
if f(x) is neither an odd nor an even function, then we must obtain expressions for a_0, a_n and b_n in full.

Lecture No -42 Examples

Example

Determine the Fourier series for the function shown.



This is neither odd nor even.

Therefore we must find a_0, a_n and b_n.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

$$f(x) = \frac{2}{\pi}x \qquad 0 < x < \pi$$

$$= 2, \qquad \pi < x < 2\pi$$

(a) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left\{\int_0^{\pi}\frac{2}{\pi}x\,dx + \int_{\pi}^{2\pi} 2\,dx\right\} = \frac{1}{\pi}\left\{\left[\frac{x^2}{\pi}\right]_0^{\pi} + \left[2x\right]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}\{\pi + 4\pi - 2\pi\}$$

$\therefore\ a_0 = 3$

(b) $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2}{\pi}x\right)\cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx + 2\int_{\pi}^{2\pi}\cos nx dx\right\} = \frac{2}{\pi}\left\{\frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + 2\int_{\pi}^{2\pi}\cos nx dx\right\}$$

$$= \frac{2}{\pi}\left[\left\{\frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{\pi n}\int_0^{\pi}\sin nx dx\right\} + \int_{\pi}^{2\pi}\cos nx dx\right] = \frac{2}{\pi}\left\{0 - \frac{1}{n\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}\right\} = \frac{2}{\pi^2 n^2}\left\{\cos n\pi - \cos 0\right\} = \frac{2}{\pi^2 n^2}\left\{\cos n\pi - 1\right\}$$

$a_n = 0$ (n even); $a_n = \frac{-4}{\pi^2 n^2}$ (n odd)

(c) To find b_n, we proceed in the same general manner

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2}{\pi}x\right)\sin nx\,dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi n}\int_0^{\pi}\cos nx dx + \int_{\pi}^{2\pi}\sin nx dx\right\} = \frac{2}{\pi}\left\{\frac{1}{\pi n}(-\pi\cos n\pi) + \frac{1}{\pi n}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{n}\cos n\pi + (0-0) - \frac{1}{n}(\cos 2\pi n - \cos n\pi)\right\} = \frac{2}{\pi}\left\{-\frac{1}{n}\cos n\pi + (0-0) - \frac{1}{n}\cos 2\pi n + \frac{1}{n}\cos n\pi\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{n}\cos 2n\pi\right\} = -\frac{2}{\pi n}\cos 2n\pi$$

But $\cos 2n\pi = 1$.

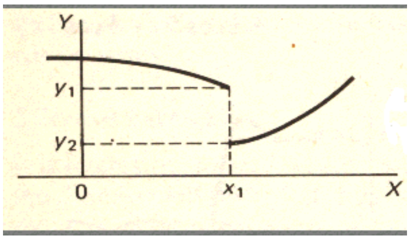
$$\therefore\ b_n = -\frac{2}{\pi n}$$

So the required series is

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\} - \frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \ldots\right\}$$

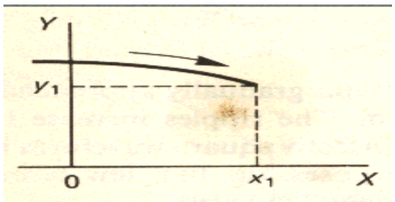
Sum of a Fourier series at a point of discontinuity

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$



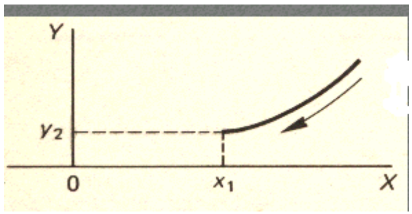
At $x = x_1$, the series converges to the value $f(x_1)$ as the number of terms including increases to infinity.

A particular point of interest occurs at a point of finite discontinuity or `jump' of the function $y = f(x)$. At $x = x_1$, the function appears to have two distinct values, y_1 and y_2.



If we approach $x = x_1$ from below that value, the limiting value of $f(x)$ is y_1.

If we approach $x = x_1$ from above that value, the limiting value of $f(x)$ is y_2.



To distinguish between these two values we write

$y_1 = f(x_1 - 0)$ denoting immediately before $x = x_1$

$y_2 = f(x_1 + 0)$

denoting immediately after $x = x_1$.

In fact, if we substitute $x = x_1$ in the Fourier series for $f(x)$, it can be shown that the series converges to the value

$\frac{1}{2}\{f(x_1 - 0) + f(x_1 + 0)\} = \frac{1}{2}(y_1 + y_2)$, the average of y_1 and y_2.

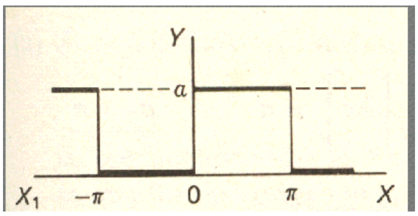
Example

Consider the function

$f(x) = 0 \qquad -\pi < x < 0$

$f(x) = a \qquad 0 < x < \pi$

$f(x) = f(x + 2\pi)$



(a) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} a\,dx$$

$$= \frac{1}{\pi}[ax]_0^{\pi} = a$$

$\therefore \quad a_0 = a$

(b) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} a \cos nx dx = \frac{a}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$$

$\therefore \quad a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{\pi} a \sin nx\, dx0 = \frac{a}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{a}{n\pi}(1 - \cos n\pi)$$

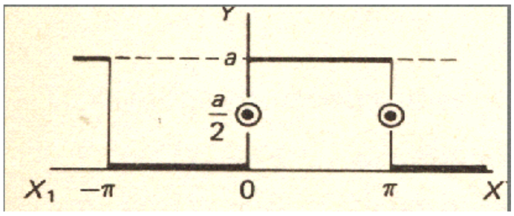
But $\cos n\pi = 1$ (n even) and $\cos n\pi = -1$ (n odd)

$b_n = 0$ (n even) and $b_n = \frac{2a}{n\pi}$ (n odd)

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a}{2} + \frac{2a}{\pi} \{\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + ...\}$$

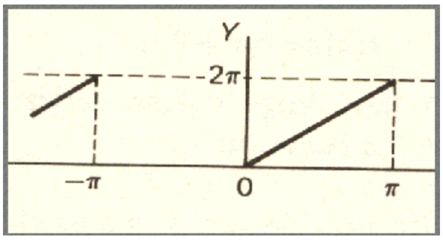
A finite discontinuity, or `jump', occurs at x = 0. If we substitute x = 0 in the series obtained, all the sine terms vanish and we get f(x) = a/2, which is, in fact, the average of the two function values at x = 0.



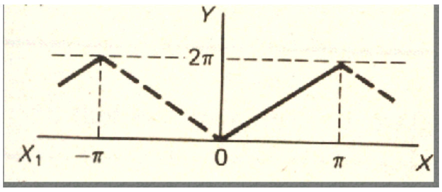
Note also that at x = π, another finite discontinuity occurs and substituting x = π in the series gives the same result. Because of this ambiguity, the function is said to be `undefined' at x = 0, x = π, etc.

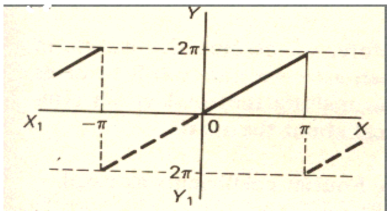
Half-Range Series

Sometime a function of period 2π is defined over the range 0 to π, instead of the normal – π to π, or 0 to 2π. We then have a choice of how to proceed. For example, if we are told between x = 0 and x = π, f(x) = 2x, then, since the period is 2π, we have no evidence of how the function behave between x = – π and x = 0.



If the waveform were as shown in (a), the function would be an even function, symmetrical about the y-axis and the series would have only cosine terms (including possibly a_0).





On the other hand, if the waveform were as shown in (b), the function would be odd, being symmetrical about the origin and the series would have only sine terms.

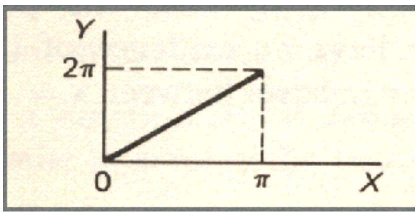
Example

A function f(x) is defined by

f(x) = 2x   0 < x < π

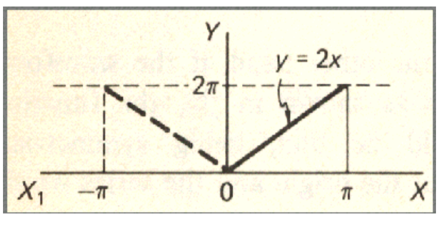
f(x) = f(x + 2π)

Obtain a half-range cosine series to represents the function.



To obtain a cosine series, i.e. a series with no sine terms, we need an even function. Therefore, we assume the waveform between x = – π and x = 0 to be as shown, making the total graph symmetrical about the y-axis.

Now we can find expressions for the Fourier coefficients as usual.



$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} 2x dx = \frac{2}{\pi}\left[x^2\right]_0^{\pi} = 2\pi$$

$\therefore\ a_0 = 2\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} 2x\cos nx\,dx \quad = \frac{4}{\pi}\int_{0}^{\pi} x\cos nx\,dx \quad = \frac{4}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_{0}^{\pi}\sin nx\,dx\right\}$$

$$= \frac{4}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_0^{\pi}\right\} = \left(\frac{4}{\pi n^2}\right)(\cos n\pi - 1)$$

$\cos n\pi = 1$ (n even) and $\cos n\pi = -1$ (n odd)

$\therefore\ a_n = 0$ (n even) and $a_n = -\dfrac{8}{\pi n^2}$ (n odd)

All that now remains is b_n which is zero, since f(x) is an even function, i.e. $b_n = 0$

So $a_0 = 2\pi$, $a_n = 0$ (n even) and $a_n = -\dfrac{8}{\pi n^2}$ (n odd),

$b_n = 0$. Therefore

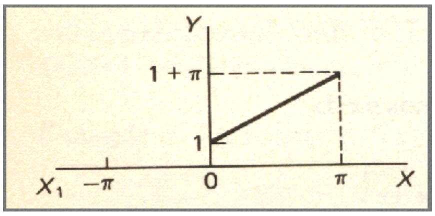
$$f(x) = \pi - \frac{8}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\}$$

Example

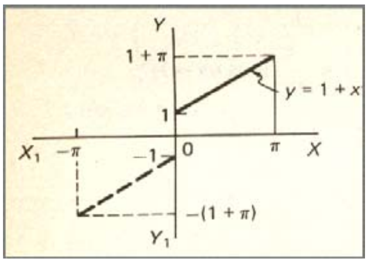
Determine a half-range sine series to repressent the function f(x) defined by

$f(x) = 1 + x \quad 0 < x < \pi$

$f(x) = f(x + 2\pi)$



We choose the waveform between $x = -\pi$ and $x = 0$ so that the graph is symmetrical about the origin. The function is then an odd function and the series will contain only sine terms.



$\therefore\ a_0 = 0$ and $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi}(1+x)\sin nx\,dx = \frac{2}{\pi}\left\{\left[(1+x)\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1+\pi}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\} = \frac{2}{\pi}\left\{\frac{1}{n} - \frac{1+\pi}{n}\cos n\pi\right\} = \frac{2}{\pi n}\{1 - (1+\pi)\cos n\pi\}$$

$\cos n\pi = 1$ (n even) and $\cos n\pi = -1$ (n odd)

$\therefore\ b_n = -\dfrac{2}{n}$ (n even)

$\therefore\ = \dfrac{4+2\pi}{\pi n}$ (n odd)

Substituting in the general expression $f(x) = \sum_{x=1}^{\infty} b_n \sin nx$ we have

$f(x)=\frac{4+2\pi}{\pi}\{\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\}-2\{\frac{1}{2}\sin 2x+\frac{1}{4}\sin 4x+\frac{1}{6}\sin 6x+....)$

and the required series obtained

$f(x)=\left(\frac{4}{\pi}+2\right)\{\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+..\}-2\{\frac{1}{2}\sin 2x+\frac{1}{4}\sin 4x+\frac{1}{6}\sin 6x+.....\}$

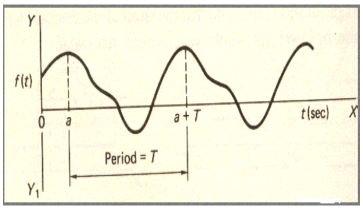
So knowledge of odd and even functions and of half-range series saves a deal of unnecessary work on occasions.

Lecture No.-43 Functions With Periods Other Than 2π

So far, we have considered functions f(x) with period 2π. In practice, we often encounter functions defined over periodic intervals other than 2π, e.g. from 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$ etc.

Functions With Period T

If y = f(x) is defined in the range $-\frac{T}{2}$ to $\frac{T}{2}$, i.e. has a period T, we can convert this to an interval of 2π by changing the units of the independent variable.



In many practical cases involving physical oscillations, the independent variable is time (t) and the periodic interval is normally denoted by T, i.e.

$$f(t) = f(t + T)$$

Each cycle is therefore completed in T seconds and the frequency f hertz (oscillations per second) of the periodic function is therefore given by $f = \frac{1}{T}$. If the angular velocity, ω radians per seconds, is defined by ω = 2πf, then

$$\omega = \frac{2\pi}{T} \text{ and } T = \frac{2\pi}{\omega}$$

The angle, x radians, at any time t is therefore x = ωt and the Fourier series to represent the function can be expressed **as**

$$\mathbf{f(t) = \frac{1}{2} a_0 + \sum_{x=1}^{\infty} \{a_n Cos\ n\omega t + b_n Sin\ n\omega t\}}$$

which can also be written in the form

$$\mathbf{f(t) = \frac{1}{2} A_0 + \sum_{x=1}^{\infty} B_n \sin(n\omega t + \phi_n)} \qquad n = 1, 2, 3, ...$$

Fourier Coefficients

With the new variable, the Fourier coefficients become:

$$f(t) = \frac{1}{2} a_0 + \sum_{x=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,dt$$

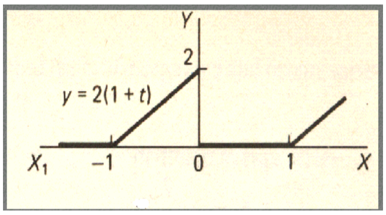
We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$, $-\frac{\pi}{\omega}$ to $\frac{\pi}{\omega}$, 0 to $\frac{2\pi}{\omega}$ etc. as is convenient, so long as they cover a complete period.

Example

Determine the Fourier series for a periodic function defined by

$f(t) = 2\ (1 + t) \qquad -1 < t < 0$
$f(t) = 0 \qquad 0 < t < 1$
$f(t) = f\ (t + 2)$

The first step is to sketch the wave which is.



$$f(t) = \frac{1}{2}\ a_0 + \sum_{x=1}^{\infty}\ \{a_n \cos n\omega t + b_n \sin n\omega t\})$$

$T = 2$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\ dt = \frac{2}{2}\int_{-1}^{1} f(t)\ dt\ = \int_{-1}^{0} 2\ (1 + t)\ dt + \int_{0}^{1}\ (0)\ dt = \left[2t + t^2\right]_1^0 = 1$$

$\therefore \qquad a_0 = 1$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t dt = \frac{2}{2}\int_{-1}^{1} f(t)\cos n\omega t\ dt = \int_{-1}^{0} 2\ (1 + t)\ \cos n\omega t\ dt$$

$$a_n = 2\left\{\left[(1+t)\frac{\sin n\omega t}{n\omega}\right]_{-1}^{0} - \frac{1}{n\omega}\int_{-1}^{0}\sin n\omega t dt\right\}\ = 2\left\{\ (0-0) - \frac{1}{n\omega}\left[-\frac{\cos n\omega t}{n\omega}\right]_{-1}^{0}\right\}$$

$$= \frac{2}{n^2\omega^2}\ (1 - \cos n\omega)$$

$$\text{Now } T = \frac{2\pi}{\omega}$$

$$\therefore\ \ \omega = \frac{2\pi}{T}\ = \frac{2\pi}{2}\ = \pi \quad \therefore\ a_n = \frac{2}{n^2\omega^2}\ (1 - \cos n\pi)$$

$\therefore \qquad a_n = 0$ (n even)

$$= \frac{4}{n^2\omega^2}\ \text{(n odd)}$$

Now for b_n

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\ \sin n\omega t\ dt$$

$$b_n = \frac{2}{2}\int_{-1}^{0} 2(1+t)\ \sin n\omega t\ dt = 2\left\{\left[(1+t)\frac{-\cos n\omega t}{n\omega}\right]_{-1}^{0} + \frac{1}{n\omega}\int_{-1}^{0}\cos n\omega t dt\right\}$$

$$= 2\left\{\ -\frac{1}{n\omega} + \frac{1}{n\omega}\left[\frac{\sin n\omega t}{n\omega}\right]_{-1}^{0}\right\} = 2\left\{-\frac{1}{n\omega} + \frac{1}{n^2\omega^2}\ (\sin n\omega)\right\}$$

As $\qquad$ before $\omega = \pi$

$$\therefore \qquad b_n = -\frac{2}{n\omega}$$

So the first few terms of the series give

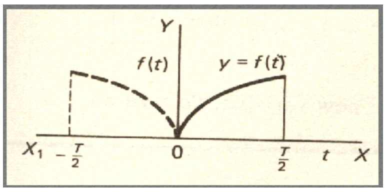
$$f(t) = \frac{1}{2} + \frac{4}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + ..\right\} - \frac{2}{\omega}\left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{4}\sin 4\omega t..\right\}$$

Half-Range Series

The theory behind the half-range sine and cosine series still applies with the new variable.

(a) Even function

Half-range cosine series



$y = f(t) \qquad 0 < t < \frac{T}{2}$

$f(t) = f(t + T)$

symmetrical about the y-axis.

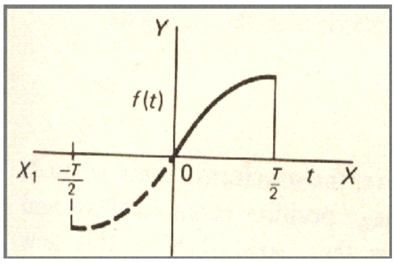
With an even function, we know that $b_n = 0$

$$\therefore \quad f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

$$\text{where } a_0 = \frac{4}{T}\int_0^{T/2} f(t)\, dt \quad \text{and} \qquad a_n = \frac{4}{T}\int_0^{T/2} f(t) \cos n\omega t\, dt$$

(b) Odd function

Half-range sine series



$y = f(t) \qquad 0 < t < \frac{T}{2}$

$f(t) = f(t + T)$

symmetrical about the origin.

$\therefore$ $a_0 = 0$ and $a_n = 0$

$$f(t) = \sum_{x=1}^{\infty} b_n \sin n\omega t;$$

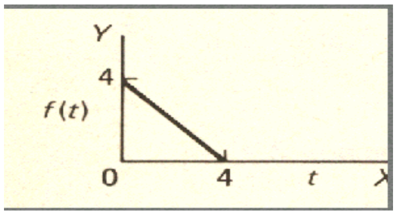
$$b_n = \frac{4}{T}\int_0^{T/2} f(t) \sin n\omega t\, dt$$

Example

A function f(t) is defined by

$f(t) = 4 - t, \qquad 0 < t < 4.$

We have to form a half-range cosine series to represent the function in this interval.



First we form an even function, i.e. symmetrical about the y-axis.

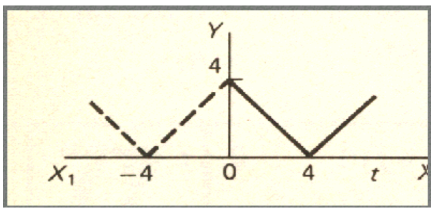
$$a_0 = \frac{4}{T}\int_0^{T/2} f_1(t)\, dt = \frac{4}{8}\int_0^{4} (4 - t)\, dt = \frac{1}{2}\int_0^{4} (4 - t)\, dt$$

$$= \frac{1}{2}\left| 4t - \frac{t^2}{2} \right|_0^4 = \frac{1}{2}\left[4(4) - \frac{(4)^2}{2} \right] = \frac{1}{2}\,[16 - 8] = \frac{1}{2}\,(8) = 4$$

$$a_n = \frac{4}{T}\int_0^{T/2} f_1(t) \cos n\omega t\, dt = \frac{4}{8}\int_0^{4} (4 - t) \cos n\omega t\, dt$$

Simple integration by parts gives

$$a_n = \frac{1}{2}\left\{ -\frac{2 \sin 4n\omega}{n\omega} - \frac{1}{n^2\omega^2}(\cos 4n\omega - 1) \right\}$$



$$\text{But } \omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\therefore \; a_n = \frac{1}{2}\left\{ -\frac{2\sin n\pi}{n\omega} - \frac{1}{n^2\omega^2}(\cos n\pi - 1) \right\}$$

$n = 1, 2, 3, \ldots\ldots$

$\sin n\pi = 0;$

$\cos n\pi = 1$ (n even); $\cos n\pi = -1$ (n odd)

$\therefore\ a_n = 0$ (n even)

$a_n = \dfrac{1}{n^2\omega^2}$ (n odd)

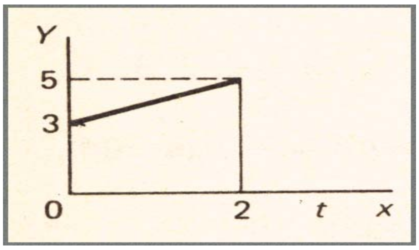
$\therefore\ f(t)=2+\dfrac{1}{\omega^2}\left\{\cos\omega t+\dfrac{1}{9}\cos 3\omega t+\dfrac{1}{25}\cos 5\omega t+\ldots\right\}$ where $\omega = \dfrac{\pi}{4}$.

Example

A function f(t) is defined by

$f(t) = 3 + t \quad 0 < t < 2$

$f(t) = f(t + 4)$

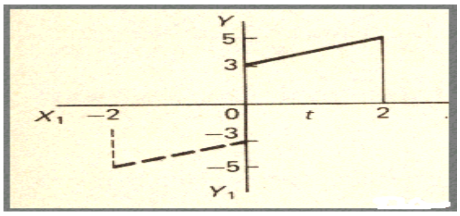


Obtain the half-range sine series for the function in this range.

Sine series required. Therefore, we form an odd function, symmetrical about the origin.

$a_0 = 0; \quad a_n = 0; \quad T = 4$

$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$



$\therefore\ b_n = \dfrac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt = \int_0^2 (3+t)\sin n\omega t\, dt$

$\int_0^2 (3 + t)\sin n\omega t\, dt = \left|\dfrac{(3+t)\cos n\omega t}{-n\omega}\right|_0^2 - \int_0^2 \dfrac{\cos n\omega t}{-n\omega}\, dt$

$= \dfrac{(3+2)\cos n\omega 2}{-n\omega} - \dfrac{3}{-n\omega} + \dfrac{1}{n\omega}\left|\dfrac{\sin n\omega t}{n\omega}\right|_0^2 = \dfrac{3}{n\omega} - \dfrac{5}{n\omega}\cos 2n\omega + \dfrac{1}{n^2\omega^2}\left[\dfrac{\sin 2n\omega}{n\omega} - 0\right]$

But $T = \dfrac{2\pi}{\omega}$ $\therefore$ $\omega = \dfrac{2\pi}{T} = \dfrac{\pi}{2}$

$= \dfrac{3}{n\omega} - \dfrac{5}{n\omega}\cos n\pi + \dfrac{1}{n^2\omega^2}\left[\dfrac{\sin n\pi}{n\omega}\right]$

$b_n = \dfrac{1}{n\omega}(3 - 5\cos 2n\omega) + \dfrac{1}{n^2\omega^2}(\sin 2n\omega)$

$\therefore\ b_n = \dfrac{1}{n\omega}(3 - 5\cos n\pi) + \dfrac{1}{n^2\omega^2}(\sin n\pi) = -\dfrac{2}{n\omega}$ (n even)

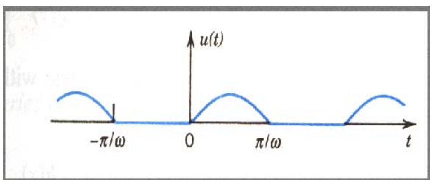
$= \dfrac{8}{n\omega}$ (n odd)

$\therefore\ f(t)=\dfrac{2}{\omega}\left\{4\sin\omega t - \dfrac{1}{2}\sin 2\omega t + \dfrac{4}{3}\sin 3\omega t - \dfrac{1}{4}\sin 4\omega t..\right\}$

Half-Wave Rectifier

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave

Find the Fourier series of the resulting periodic functions.



$u(t) = 0$ if $-T/2 < t < 0$

$= E \sin \omega t \quad 0 < t < T/2$

here $T = \dfrac{2\pi}{\omega}$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\,dt = \frac{2}{T}\int_{-T/2}^{0} 0\,dt + \frac{2}{T}\int_{0}^{T/2} E\sin\omega t\,dt = \frac{2}{T}\int_{0}^{T/2} E\sin\omega t\,dt = \frac{\omega}{\pi}\int_{0}^{\pi/\omega} E\sin\omega t\,dt$$

$$= \frac{\omega}{\pi} E\left|\frac{-\cos\omega t}{\omega}\right|_{0}^{\pi/\omega} = \frac{2E}{\pi}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\cos n\omega t\,dt = \frac{2}{T}\int_{0}^{T/2} E\sin\omega t\cos n\omega t\,dt = \frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} 2\sin\omega t\cos n\omega t\,dt$$

$$= \frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} [\sin(1+n)\omega t + \sin(1-n)\omega t]\,dt$$

If n = 1 then integral on the right is zero
and if n = 2, 3, …… then we obtain.

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} = \frac{\omega E}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)\omega} + \frac{-\cos(1-n)\pi + 1}{(1-n)\omega}\right]$$

$$= \frac{\omega E}{2\pi\omega}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)} + \frac{-\cos(1-n)\pi + 1}{(1-n)}\right]$$

if n is odd then $a_n = 0$

if n is even then $a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = \frac{E}{2\pi}\left[\frac{2 - 2n + 2 + 2n}{(1+n)(1-n)}\right] = \frac{2E}{(1-n)(1+n)\pi}$

$$= \frac{2E}{(1-n^2)\pi}$$

For b_n we have

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\sin n\omega t\,dt = \frac{2}{T}\int_{0}^{T/2} E\sin\omega t\sin n\omega t\,dt = -\frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} -2\sin\omega t\sin n\omega t\,dt$$

$$= -\frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} [\cos(1+n)\omega t - \cos(1-n)\omega t]\,dt$$

If n = 1

$$b_n = -\frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} [\cos 2\omega t - 1]\,dt = -\frac{\omega E}{2\pi}\left|\frac{\sin 2\omega t}{2\omega} - t\right|_0^{\pi/\omega} = -\frac{\omega E}{2\pi}(-\pi/\omega) = E/2$$

if $n \neq 1$

$$b_n = -\frac{\omega E}{2\pi}\int_{0}^{\pi/\omega} [\cos(1+n)\omega t - \cos(1-n)\omega t]\,dt = -\frac{\omega E}{2\pi}\left[\frac{\sin(1+n)\omega t}{(1+n)\omega} - \frac{\sin(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= -\frac{\omega E}{2\pi}\left[\frac{\sin(1+n)\pi}{(1+n)\omega} - \frac{\sin(1-n)\pi}{(1-n)\omega}\right] = 0 \qquad \text{for } n = 2, 3, 4, \ldots\ldots$$

$$u(t) = \frac{1}{2}a_0 + \sum_{n=2}^{\infty} a_n \cos n\omega t$$

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + \ldots\right)$$

Lecture No -44 Laplace Transforms

$\mathcal{L}\{e^{3t}\} = \dfrac{1}{s-3}$; $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4}$; $\mathcal{L}\{2\sin 3t + \cos 3t\} = \dfrac{s+6}{s^2+9}$

And $L^{-1}\{\dfrac{1}{s-3}\} = e^{3t}$; $L^{-1}\{\dfrac{3!}{s^4}\} = t^3$; $L^{-1}\{\dfrac{s+6}{s^2+9}\} = 2\sin 3t + \cos 3t$

We show that $L\{t^3\} = \dfrac{3!}{s^4}$ For this consider the integral

$$L(t^3) = \int_0^\infty t^3 e^{-st}dt = [t^3 \frac{e^{-st}}{-s}]_0^\infty - \int_0^\infty 3t^2 e^{-st}\frac{1}{-s}dt = \frac{-t^3}{se^{st}}\Big|_0^\infty + \frac{3}{s}\int_0^\infty t^2 e^{-st}dt$$

$$= 0 + \frac{3}{s}\{t^2 \frac{e^{-st}}{-s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}dt\} = \frac{3}{s}\frac{2}{s}\int_0^s te^{-st}dt = \frac{3.2}{s^2}\{t\frac{e^{-st}}{-s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt\}$$

$$= \frac{3.2}{s^3}\int_0^\infty e^{-st}dt = \frac{3.2}{s^3}\frac{-1}{s}\int_0^\infty e^{-st}(-s)dt = \frac{-3.2.1}{s^4}e^{-st}\Big|_0^\infty = \frac{-3.2.1}{s^4}(0-1) = \frac{3.2.1}{s^4} = \frac{3!}{s^4}$$

So that $L\{t^3\} = \dfrac{3!}{s^4}$

Laplace Transform

The Laplace Transform of a function F(t) is denoted by L{F(t)} and is defined as the integral of F(t) e^{-st} between the limits t=0 and t = ∞

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt.$$

In all cases, the constant parameter s is assumed to be positive and large enough to ensure that the product F(t) e^{-st} converges to zero as t $\rightarrow \infty$, what ever the function F(t).
In determining the transform of any function , you will appreciate that the limits are substituted for t, so that the result will be a function of s.
Laplace Transform of F(t) = a (constant).
That is

$$L(a) = \int_0^\infty ae^{-st}dt = a\int_0^\infty e^{-st}dt$$

$$= a\frac{e^{-st}}{-s}\Big|_0^\infty = \frac{-a}{s}\{0-1\} = \frac{a}{s}$$

Example

Find the laplace transform of the form e^{at} that F(t)= e^{at} where a is a constant.

$$L(e^{at}) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = [\frac{e^{-(s-a)t}}{-(s-a)}]_0^\infty$$

$$= -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right], \quad s > a$$

$$= -\frac{1}{s-a}[0-1] = \frac{1}{s-a}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

So we already have two standard transforms

$\boldsymbol{L}\{a\}=\dfrac{a}{s}$; $\quad$ $L\{e^{at}\}=\dfrac{1}{s-a}$;

$L\{4\}=\dfrac{4}{s}$; $\quad$ $L\{e^{4t}\}=\dfrac{1}{s-4}$;

$L\{-5\}=\dfrac{-5}{s}$; $\quad$ $L\{e^{-2t}\}=\dfrac{1}{s+2}$;

Laplace transform is always a function of s.

Complex Numbers Power of i

Every time a factor i^4 occurs , it can be replace by the factor 1, so that the power of i is reduced to one of the four results above.

$$i^9 = (i^4)^2 i = (1)^2 i = 1.\ i = i$$
$$i^{20} = (i^4)^5 = (1)^5 = 1$$
$$i^{30} = (i^4)^7\ i^2 = (1)^7 (-1) = 1(-1) = -1$$
$$i^{15} = (i^4)^3\ i^3 = 1(-i) = -i$$

Complex Numbers

z = 3+ 5 i , is called a complex number where 3 is real part and 5 is imaginary part of the complex number.

In general z = a+ b i , is called a complex number where a is real part and b is imaginary part of the complex number. So,

Complex Number = (Real Part)+ *i* (Imaginary Part)

Conjugate complex numbers

For a complex number a + *i* b , the complex number a - *i* b is called the conjugate of a + *i* b. Conjugate complex numbers are identical except the signs in the middle for the brackets.

- (4 + 5 i) and (4 - 5 i) are conjugate complex numbers
- (6 + 2 i) and (2 + 6 i) are not conjugate complex numbers
- (5 - 3 i) and (-5 +3 i) are not conjugate complex numbers

Remember

The product of complex number by its conjugate is always entirely real.

(3 + 4 *i*) (3 - 4 *i*) = 9 + 16 = 25

(a+ b *i*) (a - b*i*) = a^2+b^2

Euler Formaula

As we know that the series expansion of e^x ,cos x and sin x are given as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots.$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots.$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots.$$

Replace x by (*i*t) , we get

$$e^{(it)}=1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+.............$$

$$e^{(it)}=1+(it)-\frac{(t)^2}{2!}-\frac{i(t)^3}{3!}+\frac{(t)^4}{4!}+\frac{i(t)^5}{5!}-\frac{(t)^6}{6!}-\frac{i(t)^7}{7!}+.............$$

$$e^{(it)}=[1-\frac{(t)^2}{2!}+\frac{(t)^4}{4!}-\frac{(t)^6}{6!}+.............]+i[t-\frac{(t)^3}{3!}+\frac{(t)^5}{5!}-\frac{(t)^7}{7!}+.............]$$

where

$$\cos t=1-\frac{(t)^2}{2!}+\frac{(t)^4}{4!}-\frac{(t)^6}{6!}+.............$$

$$\sin t= t-\frac{(t)^3}{3!}+\frac{(t)^5}{5!}-\frac{(t)^7}{7!}+.............$$

$$e^{it}=\cos t+i\sin t$$

R(e^{it})= cost and I(e^{it}) = sin t

The Laplace transform of F(t) = sin at

$$L(\text{Sin at}) = L(I(e^{iat}))=I\int_0^\infty e^{iat}e^{-st}dt=I\int_0^\infty e^{-(s-ia)t}dt=I\left\{\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty\right\}$$

$$=I\left\{\frac{1}{-(s-ia)}(0-1)\right\}=I\left\{\frac{1}{(s-ia)}\right\}$$

$$L\{\sin at\}=I\{\frac{s+ia}{s^2+a^2}\}=I[\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}]$$

$$\therefore L\{\sin at\}=\frac{a}{s^2+a^2}$$

We can use the same method to determined to determine L{cos at }.
Since cos at is the real part of e^{iat}, written as R(e^{iat})

$$L\{\cos at\}=R\{\frac{s+ia}{s^2+a^2}\}=R[\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}]$$

$$L\{\cos at\}=\frac{s}{s^2+a^2}$$

The Transform of F(t) = t^n where n is a positive integer.

By the definition $L(t^n)=\int_0^\infty t^n e^{-st}dt$ integrating by parts

$$L(t^n)=\int_0^\infty t^n e^{-st}dt=\left[t^n\frac{e^{-st}}{-s}\right]_0^\infty+\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt=-\frac{1}{s}\left[t^n e^{-st}\right]_0^\infty+\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$\therefore\left[t^n\frac{e^{-st}}{-s}\right]_0^\infty=0-0=0 \qquad \therefore L\{t^n\}=\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \text{ -----------------(1)}$$

you will notice that $\int_0^\infty t^{n-1}e^{-st}dt$ is identical to $\int_0^\infty t^n e^{-st}dt$ except that n is replaced by (n-1)

If $\quad if\ I_n = \int_0^\infty t^n e^{-st} dt, then\, I_{n-1} = \int_0^\infty t^{n-1} e^{-st} dt$ and the result (1) becomes

$$I_n = \frac{n}{s} I_{n-1}$$

This is reduction formula and , if we now replace n by (n-1) we get

$$I_{n-1} = \frac{n-1}{s} I_{n-2}$$

If we replace n by (n-1) again in this last result , we have

$$I_{n-2} = \frac{n-2}{s} I_{n-3}$$

$$So\ \ I_n = \int_0^\infty t^n e^{-st} dt = \frac{n}{s} I_{n-1} = \frac{n}{s}.\frac{n-1}{s} I_{n-2} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s} I_{n-3} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s} I_{n-4}$$

So finally, we have

$$I_n = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}.\frac{n-4}{s}..............\frac{2}{s}.\frac{1}{s} I_0$$

But

$$I_0 = L\{t^0\} = L\{1\} = \frac{1}{s}$$

$$I_n = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\frac{n-3}{s}\frac{n-4}{s}..............\frac{2}{s}\frac{1}{s} I_0 = \frac{n!}{s^{n+1}}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{t\} = \frac{1}{s^2}; L\{t^2\} = \frac{2!}{s^3}; L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

Laplace Transform of F(t) = sinh at and F(t) = cosh at.

Starting from the exponential definitions of sinh at and cosh at

$$i.e. \quad \sinh at = \frac{1}{2}(e^{at} - e^{-at})\ and\ \cosh at = \frac{1}{2}(e^{at} + e^{-at})$$

We proceed as follow

$a) \quad F(t) = \sinh at$

$$\text{Sinh at} = \int_0^\infty \sinh at\ e^{-st} dt = \frac{1}{2}\int_0^\infty (e^{at} - e^{-at}) e^{-st} dt = \frac{1}{2}\int_0^\infty (e^{at}e^{-st} - e^{-at}e^{-st}) dt = \frac{1}{2}\int_0^\infty (e^{-(s-a)t} - e^{-(s+a)t}) dt$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty - \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right\} = \frac{1}{2}\left\{\frac{1}{-(s-a)}(0-1) - \frac{1}{-(s+a)}(0-1)\right\} = \frac{1}{2}\left\{\frac{1}{(s-a)} - \frac{1}{(s+a)}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+a-s+a}{(s-a)(s+a)}\right\} = \frac{1}{2}\left\{\frac{2a}{s^2-a^2}\right\} = \left\{\frac{a}{s^2-a^2}\right\}$$

$b) \quad F(t) = \cosh at$

$$\text{L(cosh at)} = \text{L}\left(\frac{1}{2}\left(e^{at} + e^{-at}\right)\right) = \frac{1}{2}\int_0^\infty (e^{at} + e^{at})e^{-st}dt = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}dt + \int_0^\infty e^{-(s+a)t}dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right\} = \frac{1}{2}\left\{\frac{1}{-(s-a)}(0-1) + \frac{1}{-(s+a)}(0-1)\right\} = \frac{1}{2}\left\{\frac{1}{(s-a)} + \frac{1}{(s+a)}\right\}$$

$$= \frac{1}{2}\left\{\frac{s+a+s-a}{(s-a)(s+a)}\right\} = \frac{1}{2}\left\{\frac{2s}{s^2-a^2}\right\} = \left\{\frac{s}{s^2-a^2}\right\}$$

$$\text{L(cosh at)} = \left\{\frac{s}{s^2-a^2}\right\}$$

Several Standard Results

$$L\{a\} = \frac{a}{s}; \qquad L\{e^{at}\} = \frac{1}{s-a}; \qquad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}; \qquad L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2}; \qquad L\{\cosh at\} = \frac{s}{s^2-a^2}$$

We can , of course, combine these transforms by adding or subtracting as necessary , but they must not be multiplied together to form the transform of a product.

Example

$$a)\, L\{2\sin 3t + \cos 3t\} = 2L(\sin 3t) + L(\cos 3t) = 2.\frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+6}{s^2+9}$$

$$b)\, L\{4e^{2t} + 3\cosh 4t\} = 4L(e^{2t}) + L(3\cosh 4t) = 4.\frac{1}{s-2} + 3.\frac{s}{s^2-16} = \frac{4}{s-2} + \frac{3s}{s^2-16}$$

$$= \frac{7s^2-6s-64}{(s-2)(s^2-16)}$$

Lecture No -45 Theorems

Theorem 1
The First Shift Theorem

The first Shift theorem states that if **L{F(t)}= f(s)** then **L{$e^{(-at)}$ F(t)}= f(s+a)**
The transform **L{$e^{(-at)}$ F(t)}** is thus the same as **L{F(t)}** with s everywhere in the result replaced by (s+a)

Example

$$L\{\sin 2t\}=\frac{2}{s^2+4} \quad \textbf{then} \quad L\{e^{-3t}\sin 2t\}=\frac{2}{(s+3)^2+4}=\frac{2}{s^2+6s+13}$$

Example

$L\{t^2\}=\dfrac{2}{s^3}$; $L\{t^2e^{4t}\}$ is the same with s replaced by (s-4)

So $L\{t^2e^{4t}\}=\dfrac{2}{(s-4)^3}$

Theorem 2
Multiplying by t

If L{F(t)}= f(s) **then** $L\{t(F(t)\}=-\dfrac{d}{ds}\{f(s)\}$

Example $L\{\sin 2t\}=\dfrac{2}{s^2+4}$

And $L\{t\sin 2t\} = -\dfrac{d}{ds}\left[\dfrac{2}{(s^2+4)}\right]=\dfrac{4s}{(s^2+4)^2}$

Example $L\{t\cos 3t\}=-\dfrac{d}{ds}\left(\dfrac{s}{s^2-9}\right)=-\dfrac{(s^2-9)-s(2s)}{(s^2-9)^2}=-\dfrac{s^2-9-2s^2}{(s^2-9)^2}=\dfrac{s^2+9}{(s^2-9)^2}$

We could, if necessary, take this a stage further and find

$$L\{t\cos 3t\}=-\frac{d}{ds}\left(\frac{s^2+9}{(s^2-9)^2}\right)=\frac{2s(s^2+27)}{(s^2-9)^3}$$

Theorem Obviously extends the range of function that we can deal with. So, in general

If L{F(t)}= f(s) **then** $L\{t^n(F(t)\}=(-1)^n\dfrac{d^n}{ds^n}\{f(s)\}$

Theorem
Dividing by t

If L{F(t)}= f(s) **then** $L\{\dfrac{F(t)}{t}\}=\displaystyle\int_s^{\infty} f(s)ds$

Example Determine $L\{\dfrac{\sin at}{t}\}$

$As\ L\{\sin at\}=\dfrac{a}{s^2+a^2}$

$$\therefore L\{\frac{\sin at}{t}\}=\int_s^{\infty}\frac{a}{s^2+a^2}ds=\left[\tan^{-1}(\frac{s}{a})\right]_s^{\infty}=\frac{\pi}{2}-\tan^{-1}(\frac{s}{a})=\tan^{-1}(\frac{a}{s})$$

Example Determine $L\{\frac{\text{1-cos 2t}}{t}\}$

As $L\{\text{1-cos 2t}\}=\frac{1}{s}-\frac{s}{s^2+4}$

Then by Theorem 3,

$$L\{\frac{\text{1-cos 2t}}{t}\} = \int_s^\infty (\frac{1}{s}-\frac{s}{s^2+4})\,ds = \left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty = \left[\frac{1}{2}.2\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \left[\frac{1}{2}\ln s^2 - \frac{1}{2}\ln(s^2+4)\right]_s^\infty = \frac{1}{2}\left[\ln s - \ln(s^2+4)\right]_s^\infty = \frac{1}{2}\left[\ln\frac{s^2}{(s^2+4)}\right]_s^\infty$$

$$\textit{When } s \to \infty \textit{ then } \ln\frac{s^2}{(s^2+4)} \to \ln 1 = 0$$

$$L\{\frac{\text{1-cos 2t}}{t}\} = \int_s^\infty (\frac{1}{s}-\frac{s}{s^2+4})\,ds = -\frac{1}{2}\left[\ln\frac{s^2}{(s^2+4)}\right] = \ln\left[\frac{s^2}{(s^2+4)}\right]^{-\frac{1}{2}} = \ln\sqrt{\frac{s^2+4}{s^2}}$$

Standard Forms

F(t)	$\mathcal{L}$ {F(t)} = f(s)
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
sin at	$\frac{a}{s^2+a^2}$
cos at	$\frac{a}{s^2+a^2}$
sinh at	$\frac{a}{s^2-a^2}$
cosh at	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$

(n a positive integer)

Theorem 1 The First Shift Theorem

If L{F(t)}= f(s) then **L{$e^{(-at)}$ F(t)}= f(s+a)**

Theorem 2 Multiplying by t

If L{F(t)}= f(s) **then** $L\{t(F(t)\} = -\frac{d}{ds}\{f(s)\}$

Theorem 3 Dividing by t

If L{F(t)}= f(s) **then** $L\{\frac{F(t)}{t}\} = \int_s^\infty f(s)ds$

Provided $\lim_{t\to 0}\left\{\frac{F(t)}{t}\right\}$ *exists.*

Inverse Transforms

Here we have the reverse process i.e. given a Laplace transform, we have to find the function of t to which it belongs. For example, we know that $\frac{a}{s^2+a^2}$ is the Laplace Transform of **sin at** , so we can now write $L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$, the symbol L^{-1} indicating the inverse transform and not a reciprocal.

$(a)\qquad L^{-1}\left\{\frac{1}{s-2}\right\}=e^{-2t}$

$(b)\qquad L^{-1}\left\{\frac{s}{s^2+25}\right\}=\cos 5t$

$(c)\qquad L^{-1}\left\{\frac{4}{s}\right\}=4$

$(d)\qquad L^{-1}\left\{\frac{12}{s^2-9}\right\}=4\sinh 3t$

But what about $L^{-1}\left\{\frac{3s}{s^2-s-6}\right\}$, it happens that we can write $\frac{3s}{s^2-s-6}$ as the sum of two simpler functions $\frac{1}{s+2}+\frac{1}{s-3}$ which , of course , makes all the difference , since we can now proceed. $L^{-1}\left\{\frac{3s}{s^2-s-6}\right\}=L^{-1}\left\{\frac{1}{s+2}\right\}+L^{-1}\left\{\frac{1}{s-3}\right\}=e^{-2t}+2e^{3t}$

Rules of Partial Fractions

1. The numerator must be of lower degree than denominator. If it is not , then we first divide out.
2. Factorise the denominator into its prime factors. These determine the shapes of the partial fraction.
3. A linear factor (s+a) gives a partial fraction $\frac{A}{s+a}$ is a constant to be determined.
4. A repeated factor $(s+a)^2$ gives $\frac{A}{s+a}+\frac{B}{(s+a)^2}$
5. Similarly $(s+a)^3$ gives $\frac{A}{s+a}+\frac{B}{(s+a)^2}+\frac{C}{(s+a)^3}$
6. A quadratic Factor (s^2+ps+q) gives $\frac{Ps+Q}{s^2+ps+q}$
7. Repeated quadratic Factor $(s^2+ps+q)^2$ gives $\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}$

So $\frac{s-19}{(s-2)(s+5)}$ has partial fraction of the form $\frac{A}{(s-2)}+\frac{B}{(s+5)}$ and $\frac{3s^2-4s+11}{(s+3)(s-2)^2}$ has partial fraction $\frac{A}{(s+3)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}$

Example

To determine $L^{-1}\left\{\dfrac{5s+1}{s^2-s-12}\right\}$

a) First we check that the numerator is of lower degree than the denominator. In fact this is so.
b) Factories the denominator

$$\frac{5s+1}{s^2-s-12}=\frac{5s+1}{(s-4)(s+3)}=\frac{A}{(s-4)}+\frac{B}{(s+3)}$$

We therefore have an identity

$$\frac{5s+1}{s^2-s-12}=\frac{A}{(s-4)}+\frac{B}{(s+3)}$$

which is true for any value of s we care to substitute
If we multiply through by the denominator (s^2-s-12) we have

$5s + 1 \equiv A(s + 3) + B(s - 4)$

We now substitute covenant values for s

i) Let (s -4) =0 that is s = 4 therefore 21 = A(7) + B(0) $\Rightarrow$ A= 3
ii) Let (s + 3) =0 that is s = -3 therefore B = 2

So $$\frac{5s+1}{s^2-s-12}=\frac{3}{(s-4)}+\frac{2}{(s+3)}$$

$$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}=3e^{4t}+2e^{-3t}$$

Example Determined $L^{-1}\left\{\dfrac{9s-8}{s^2-2s}\right\}$

$$L\{F(t)\}=\frac{9s-8}{s^2-2s}$$

a) Numerator of first degree ; denominator of second degree.
b) $\dfrac{9s-8}{s^2-2s}=\dfrac{A}{s}+\dfrac{B}{s-2}$
c) Multiply by s(s – 2)
$\therefore 9s - 8 \equiv A(s - 2) + B(s)$
d) Put s = 0
$\therefore -8 \equiv A(-2) + B(0)$ $\therefore A = 4$
e) Put s – 2 , i.e. s = 2

$\therefore$ 10 = A (0) + B (0) $\therefore$ B = 5

$$\therefore F(t) = L^{-1}\left\{\frac{4}{s}+\frac{5}{s-2}\right\} = 4 + 5e^{2t}$$

Table of inverse transforms

Standard transforms

f(s)	F(t)
$\frac{a}{s}$	a
$\frac{1}{s+a}$	e^{-at}
$\frac{n!}{s^{n+1}}$	t^n (n a positive integer)
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$ (n a positive integer)
$\frac{a}{s^2+a^2}$	sin at
$\frac{s}{s^2+a^2}$	Cos at
$\frac{a}{s^2-a^2}$	Sinh at
$\frac{s}{s^2-a^2}$	Cos ax

Transforms Of Derivatives

Let $F^/(t)$ denote the first derivative of F(t) with respect to t , $F^{//}(t)$ denote the second derivative of F(t) with respect to t, etc.

Then $L\{F^/(t)\}=\int_0^\infty e^{-st}F^/(t)dt$ by definition ,

Integrating By Parts

$$L\{F^/(t)\}=\left[e^{-st}F(t)\right]_0^\infty-\int_0^\infty F(t)(-se^{-st})dt$$

when $t\to 0, e^{-st}F(t)\to 0$

$$L\{F^/(t)\}=-F(0)+s\int_0^\infty e^{-st}F(t)dt$$

$$L\{F^/(t)\}=-F(0)+sL\{F(t)\}$$

$$L\{F^{//}(t)\}=-F^/(0)+sL\{F^/(t)\}=-F^/(0)+s\left[-F(0)+sL\{F(t)\}\right]$$

$$L\{F^{//}(t)\}=s^2L\{F(t)\}-sF(0)+F^/(0)$$

$$L\{F^{///}(t)\}=s^3L\{F(t)\}-s^2F(0)-sF^/(0)-F^{//}(0)$$

$$L\{F^{iv}(t)\}=s^4L\{F(t)\}-s^3F(0)-s^2F^/(0)-sF^{//}(0)-F^{///}(0)$$

Differential Equation And Its Solution

$\frac{dx}{dt} - 2x = 4$ -----------------(1)

Its Solution is x = -2 + 3e^{2t} , To verify it we find $\frac{dx}{dt}$

$\frac{dx}{dt} = 6e^{2t}$ then

$\frac{dx}{dt} - 2x = 6e^{2t} - 2(-2 + 3e^{2t})$

$= 6e^{2t} + 4 - 6e^{2t} = 4$

So equation (1) is satisfied. Hence x = -2 + 3e^{2t} is solution of $\frac{dx}{dt} - 2x = 4$

Example Solve the differential equation $\frac{dx}{dt} - 2x = 4$ given that at t = 0 , x = 1

Taking Laplace transform as

$$L\left[\frac{d}{dt}(x(t)) - 2L(x(t))\right] = L(4) \Rightarrow L\left[\frac{d}{dt}(x(t))\right] - 2L(x(t)) = L(4) \Rightarrow sL(x(t)) - x(0) - 2L(x(t)) = \frac{4}{s}$$

$$(s-2)L(x(t)) - x(0) = \frac{4}{s} \Rightarrow (s-2)L(x(t)) - 1 = \frac{4}{s} \Rightarrow (s-2)L(x(t)) = \frac{4}{s} + 1 \Rightarrow (s-2)L(x(t)) = \frac{4+s}{s}$$

$$\Rightarrow L(x(t)) = \frac{4+s}{s(s-2)} \Rightarrow x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] \text{-----------------------(1)}$$

First we do the partial fraction of $\frac{4+s}{s(s-2)}$

$$\frac{4+s}{s(s-2)} = \frac{A}{s} + \frac{B}{(s-2)}$$

$$\Rightarrow 4 + s = A(s-2) + B(s) \text{ -----------------(2)}$$

Put s = 0 in equation (2) ; 4 = -2A ; A = -2

Put s = 2 in equation (2) ; 6 = B(2) ; B = 3

$$So\ \frac{4+s}{s(s-2)} = \frac{-2}{s} + \frac{3}{s-2}$$

Equation # (1) becomes

$$x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] = L^{-1}\left[\frac{-2}{s}\right] + L^{-1}\left[\frac{3}{s-2}\right] = -2 + 3e^{2t}$$

Solution of differential equation by laplace transforms

To solve a differential equation by Laplace transforms, we go through Laplace transforms, we go through four distinct stages.

(a) Re- write the equation in term of Laplace transforms.
(b) Insert the given initial conditions.
(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

Solve the equation

$\frac{d^2x}{dt^2} - 3\frac{d\ x}{dt} + 2x = 2e^{3t}$ given that at t = 0 , x = 5 and $\frac{dx}{dt} = 7$

$x^{//}(t) - 3x^{/}(t) + 2x(t) = 2e^{3t}$

Given x (0) = 5 , $x^{/}$ (0) = 7

$L\left(x^{//}(t)\right) - 3L(x^{/}(t)) + 2L(x(t)) = 2L(e^{3t})$

$$s^2L\{x(t)\} - s\,x(0) - x^{/}(0) - 3\{s\ L(x(t)) - x(0)\} + 2L\,x(t) = \frac{2}{s-3}$$

We rewrite the equation in term of its transforms.

$$L\left[\frac{d^2x}{dt^2}\right] - 3L\left[\frac{d\ x}{dt}\right] + 2L\left[x\right] = 2L\left[e^{3t}\right]$$

$$\left[[s^2L(x(t)) - s\,x(0) - x^{/}(0)] - 3[s\,L(x(t)) - x(0)\,] + 2L(e^{3t})\right]$$

At t = 0 , x = 5 , $\frac{dx}{dt} = 7$

So x (0) = 5 , $x^{/}$(0) = 7

$$s^2L(x(t)) - s(5) - 7 - 3\{sL(x(t)) + 3(5)\} + 2L(x(t)) = \frac{2}{s-3}$$

$$s^2L(x(t)) - 3sL(x(t)) + 2L(x(t)) = \frac{2}{s-3} - 8 + 5s$$

$$(s^2 - 3s + 2)L(x(t)) = \frac{2 - 8s + 24 + 5s^2 - 15s}{s-3}$$

$$L(x(t)) = \frac{2 - 8s + 24 + 5s^2 - 15s}{(s-1)(s-2)(s-3)} = \frac{5s^2 - 23s + 24}{(s-1)(s-2)(s-3)}$$

Making Partial fraction of R.H.S , We have

$$L(x(t)) = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

After solving these we get A= 3 , B = 2 and C = 0

So $L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)} + \frac{0}{(s-3)}$

$$L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)}$$

$$x(t) = L^{-1}\{\frac{3}{(s-1)}\} + L^{-1}\{\frac{2}{(s-2)}\} = 3e^{t} + 2e^{2t}$$