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High-precision calculation of the 4-loop contribution to the electron $g-2$ in QED

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Abstract

I have evaluated up to 1100 digits of precision the contribution of the 891 4-loop Feynman diagrams contributing to the electron $g-2$ in QED. The total mass-independent 4-loop contribution is

$$a_e = -1.912245764926445574152647167439830054060873390658725345\dots \left(\frac{\alpha}{\pi}\right)^4.$$

I have fit a semi-analytical expression to the numerical value. The expression contains harmonic polylogarithms of argument $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$, one-dimensional integrals of products of complete elliptic integrals and six finite parts of master integrals, evaluated up to 4800 digits.

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I have evaluated up to 1100 digits of precision the mass-independent contribution to the electron $g-2$ anomaly of all the 891 diagrams in 4-loop QED, thus finalizing a twenty-year effort [1–7] begun after the completion of the calculation of 3-loop QED contribution [8].

Having extracted the power of the fine structure constant α

$$a_e(4\text{-loop}) = a_e^{(4)} \left(\frac{\alpha}{\pi} \right)^4 , \quad (1)$$

the first digits of the result are

$$a_e^{(4)} = -1.912245764926445574152647167439830054060873390658725345171329848\dots . \quad (2)$$

The full-precision result is shown in table 1. The result (2) is in excellent agreement (0.9σ) with the numerical value

$$a_e^{(4)}(\text{Ref. [18]}) = -1.91298(84) , \quad (3)$$

latest result of a really impressive pluridecennial effort [9–18].

By using the best numerical value of $a_e(5\text{-loop}) = 7.795(336) \left(\frac{\alpha}{\pi} \right)^5$ (Ref. [18]), the measurement of the fine structure constant [19]

$$\alpha^{-1} = 137.035\,999\,040(90) ,$$

and the values of mass-dependent QED, hadronic and electroweak contributions (see Ref. [18] and references therein), one finds

$$a_e^{\text{th}} = 1\,159\,652\,181.664(23)(16)(763) \times 10^{-12} , \quad (4)$$

where the first error comes from $a_e^{(5)}$, the second one from the hadronic and electroweak corrections, the last one from α . Conversely, using the experimental measurement of a_e [20]

$$a_e^{\text{exp}} = 1\,159\,652\,180.73(0.28) \times 10^{-12} ,$$

one finds

$$\alpha^{-1}(a_e) = 137.035\,999\,1596(27)(18)(331) ,$$

where the errors come respectively from $a_e(5\text{-loop})$, hadronic and electroweak corrections, and a_e .

The 891 vertex diagrams contributing to $a_e^{(4)}$ are not shown for reasons of space. They can be obtained by inserting an external photon in each possible electron line of the 104 4-loop self-mass diagrams shown in Fig.1, excluding the vertex diagrams with closed electron loops with an odd number of vertices which give null contribution because of the Furry's theorem. The vertex diagrams can be arranged in 25 gauge-invariant sets (Fig.2), classifying them according to the number of photon corrections on the same side of the main electron line and the insertions of electron loops (see Ref. [21] for more details on the 3-loop classification). The numerical contributions of each set, truncated to 40 digits, are listed in the table 2. Adding respectively the contributions of diagrams with and without closed electron loops one finds

$$a_e^{(4)}(\text{no closed electron loops}) = -2.176866027739540077443259355895893938670 , \quad (5)$$

$$a_e^{(4)}(\text{closed electron loops only}) = 0.264620262813094503290612188456063884609 . \quad (6)$$

The contributions of the sets 17 and 18, the sum of contributions of the sets 11 and 12, and the sum of the contributions of the sets 15 and 16 are in perfect agreement with the analytical results of Ref. [22].

The contributions of all diagrams can be expressed by means of 334 master integrals belonging to 220 topologies. I have fit analytical expressions to the high-precision numerical values of all master integrals and diagram contributions by using the PSLQ algorithm [23,24]. The analytical expression of $a_e^{(4)}$ contains values of harmonic polylogarithms [25] with argument 1, $\frac{1}{2}$, $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$, a family of one-dimensional integrals of products of elliptic integrals, and the finite terms of the ϵ -expansions of six master integrals belonging to the topologies 81 and 83 of Fig.1. Work is still in progress to fit analytically these six unknown elliptical constants. The result of the analytical fit can be written as follows:

$$\begin{aligned} a_e^{(4)} = & T_0 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + \sqrt{3}(V_{4a} + V_{6a}) + V_{6b} + V_{7b} + W_{6b} + W_{7b} \\ & + \sqrt{3}(E_{4a} + E_{5a} + E_{6a} + E_{7a}) + E_{6b} + E_{7b} + U. \end{aligned} \quad (7)$$

The terms have been arranged in blocks with equal transcendental weight. The index number is the weight. The terms containing the “usual” transcendental constants are:

$$T_0 + T_2 + T_3 = \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2, \quad (8)$$

$$T_4 = \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right), \quad (9)$$

$$\begin{aligned} T_5 = & -\frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) - \frac{221581}{2160}\zeta(4)\ln 2 \\ & + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right), \end{aligned} \quad (10)$$

$$\begin{aligned} T_6 = & \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 \\ & - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 \\ & - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2, \end{aligned} \quad (11)$$

$$\begin{aligned}
T_7 = & \frac{2895304273}{435456} \zeta(7) + \frac{670276309}{193536} \zeta(4)\zeta(3) + \frac{85933}{63} a_4 \zeta(3) + \frac{7121162687}{967680} \zeta(5)\zeta(2) \\
& - \frac{142793}{18} a_5 \zeta(2) - \frac{195848}{21} a_7 + \frac{195848}{63} b_7 - \frac{116506}{189} d_7 \\
& - \frac{4136495}{384} \zeta(6) \ln 2 - \frac{1053568}{189} a_6 \ln 2 + \frac{233012}{189} b_6 \ln 2 + \frac{407771}{432} \zeta^2(3) \ln 2 \\
& - \frac{8937}{2} a_4 \zeta(2) \ln 2 + \frac{833683}{3024} \zeta(5) \ln^2 2 - \frac{3995099}{6048} \zeta(3)\zeta(2) \ln^2 2 - \frac{233012}{189} a_5 \ln^2 2 \\
& + \frac{1705273}{1512} \zeta(4) \ln^3 2 + \frac{602303}{4536} \zeta(3) \ln^4 2 - \frac{1650461}{11340} \zeta(2) \ln^5 2 + \frac{52177}{15876} \ln^7 2 .
\end{aligned} \tag{12}$$

The terms containing harmonic polylogarithms of $e^{i\pi/3}$, $e^{2i\pi/3}$:

$$V_{4a} = -\frac{14101}{480} \text{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440} \zeta(2) \text{Cl}_2\left(\frac{\pi}{3}\right), \tag{13}$$

$$\begin{aligned}
V_{6a} = & \frac{494}{27} \text{Im}H_{0,0,0,1,-1,-1}\left(e^{i\pi/3}\right) + \frac{494}{27} \text{Im}H_{0,0,0,1,-1,1}\left(e^{i2\pi/3}\right) + \frac{494}{27} \text{Im}H_{0,0,0,1,1,-1}\left(e^{i2\pi/3}\right) \\
& + 19 \text{Im}H_{0,0,1,0,1,1}\left(e^{i2\pi/3}\right) + \frac{437}{12} \text{Im}H_{0,0,0,1,1,1}\left(e^{i2\pi/3}\right) + \frac{29812}{297} \text{Cl}_6\left(\frac{\pi}{3}\right) \\
& + \frac{4940}{81} a_4 \text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984} \zeta(5)\pi - \frac{129251}{81} \zeta(4) \text{Cl}_2\left(\frac{\pi}{3}\right) \\
& - \frac{892}{15} \text{Im}H_{0,1,1,-1}\left(e^{i2\pi/3}\right) \zeta(2) - \frac{1784}{45} \text{Im}H_{0,1,1,-1}\left(e^{i\pi/3}\right) \zeta(2) + \frac{1729}{54} \zeta(3) \text{Im}H_{0,1,-1}\left(e^{i\pi/3}\right) \\
& + \frac{1729}{36} \zeta(3) \text{Im}H_{0,1,1}\left(e^{i2\pi/3}\right) + \frac{837190}{729} \text{Cl}_4\left(\frac{\pi}{3}\right) \zeta(2) + \frac{25937}{4860} \zeta(3) \zeta(2) \pi
\end{aligned} \tag{14}$$

$$\begin{aligned}
& - \frac{223}{243} \zeta(4) \pi \ln 2 + \frac{892}{9} \text{Im}H_{0,1,-1}\left(e^{i\pi/3}\right) \zeta(2) \ln 2 + \frac{446}{3} \text{Im}H_{0,1,1}\left(e^{i2\pi/3}\right) \zeta(2) \ln 2 \\
& - \frac{7925}{81} \text{Cl}_2\left(\frac{\pi}{3}\right) \zeta(2) \ln^2 2 + \frac{1235}{486} \text{Cl}_2\left(\frac{\pi}{3}\right) \ln^4 2 ,
\end{aligned}$$

$$V_{6b} = \frac{13487}{60} \text{Re}H_{0,0,0,1,0,1}\left(e^{i\pi/3}\right) + \frac{13487}{60} \text{Cl}_4\left(\frac{\pi}{3}\right) \text{Cl}_2\left(\frac{\pi}{3}\right) + \frac{136781}{360} \text{Cl}_2^2\left(\frac{\pi}{3}\right) \zeta(2), \tag{15}$$

$$\begin{aligned}
V_{7b} = & \frac{651}{4} \operatorname{Re} H_{0,0,0,1,0,1,-1} \left(e^{i\frac{\pi}{3}} \right) + 651 \operatorname{Re} H_{0,0,0,0,1,1,-1} \left(e^{i\frac{\pi}{3}} \right) - \frac{17577}{32} \operatorname{Re} H_{0,0,1,0,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \\
& - \frac{87885}{64} \operatorname{Re} H_{0,0,0,1,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) - \frac{17577}{8} \operatorname{Re} H_{0,0,0,0,1,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{651}{4} \operatorname{Cl}_4 \left(\frac{\pi}{3} \right) \operatorname{Im} H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \\
& + \frac{1953}{8} \operatorname{Cl}_4 \left(\frac{\pi}{3} \right) \operatorname{Im} H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) + \frac{31465}{176} \operatorname{Cl}_6 \left(\frac{\pi}{3} \right) \pi + \frac{211}{4} \operatorname{Re} H_{0,1,0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) \\
& + \frac{211}{2} \operatorname{Re} H_{0,0,1,1,-1} \left(e^{i\frac{\pi}{3}} \right) \zeta(2) + \frac{1899}{16} \operatorname{Re} H_{0,1,0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) + \frac{1899}{8} \operatorname{Re} H_{0,0,1,1,1} \left(e^{i\frac{2\pi}{3}} \right) \zeta(2) \\
& + \frac{211}{4} \operatorname{Im} H_{0,1,-1} \left(e^{i\frac{\pi}{3}} \right) \operatorname{Cl}_2 \left(\frac{\pi}{3} \right) \zeta(2) + \frac{633}{8} \operatorname{Im} H_{0,1,1} \left(e^{i\frac{2\pi}{3}} \right) \operatorname{Cl}_2 \left(\frac{\pi}{3} \right) \zeta(2) .
\end{aligned} \tag{16}$$

The terms containing harmonic polylogarithms of $e^{\frac{i\pi}{2}}$:

$$W_{6b} = -\frac{28276}{25} \zeta(2) \operatorname{Cl}_2 \left(\frac{\pi}{2} \right)^2 , \tag{17}$$

$$\begin{aligned}
W_{7b} = & 104 \left(4 \operatorname{Re} H_{0,1,0,1,1} \left(e^{i\frac{\pi}{2}} \right) \zeta(2) + 4 \operatorname{Im} H_{0,1,1} \left(e^{i\frac{\pi}{2}} \right) \operatorname{Cl}_2 \left(\frac{\pi}{2} \right) \zeta(2) - 2 \operatorname{Cl}_4 \left(\frac{\pi}{2} \right) \zeta(2) \pi \right. \\
& \left. + \operatorname{Cl}_2^2 \left(\frac{\pi}{2} \right) \zeta(2) \ln 2 \right) .
\end{aligned} \tag{18}$$

The terms containing elliptic constants:

$$E_{4a} = \pi \left(-\frac{28458503}{691200} B_3 + \frac{250077961}{18662400} C_3 \right) , \tag{19}$$

$$E_{5a} = \frac{483913}{77760} \pi f_2(0, 0, 1) , \tag{20}$$

$$E_{6a} = \pi \left(\frac{4715}{1944} \ln 2 f_2(0, 0, 1) + \frac{270433}{10935} f_2(0, 2, 0) - \frac{188147}{4860} f_2(0, 1, 1) + \frac{188147}{12960} f_2(0, 0, 2) \right) , \tag{21}$$

$$E_{6b} = -\frac{4715}{1458} \zeta(2) f_1(0, 0, 1) , \tag{22}$$

$$\begin{aligned}
E_{7a} = & \pi \left(\frac{826595}{248832} \zeta(2) f_2(0, 0, 1) - \frac{5525}{432} \ln 2 f_2(0, 0, 2) + \frac{5525}{162} \ln 2 f_2(0, 1, 1) \right. \\
& - \frac{5525}{243} \ln 2 f_2(0, 2, 0) + \frac{526015}{248832} f_2(0, 0, 3) - \frac{4675}{768} f_2(0, 1, 2) + \frac{1805965}{248832} f_2(0, 2, 1) \\
& - \frac{3710675}{1119744} f_2(0, 3, 0) - \frac{75145}{124416} f_2(1, 0, 2) - \frac{213635}{124416} f_2(1, 1, 1) + \frac{168455}{62208} f_2(1, 2, 0) \quad (23) \\
& \left. + \frac{75145}{248832} f_2(2, 0, 1) + \frac{69245}{124416} f_2(2, 1, 0) \right) , \\
E_{7b} = & \zeta(2) \left(\frac{2541575}{82944} f_1(0, 0, 2) - \frac{556445}{6912} f_1(0, 1, 1) + \frac{54515}{972} f_1(0, 2, 0) - \frac{75145}{20736} f_1(1, 0, 1) \right) . \quad (24)
\end{aligned}$$

The term containing the ϵ^0 coefficients of the ϵ -expansion of six master integrals (see f, f', f'', g, g', g'' of Fig.3):

$$U = -\frac{541}{300} C_{81a} - \frac{629}{60} C_{81b} + \frac{49}{3} C_{81c} - \frac{327}{160} C_{83a} + \frac{49}{36} C_{83b} + \frac{37}{6} C_{83c} . \quad (25)$$

The numerical values of Eqs.(8)-(25) are listed in Table 3. In the above expressions $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$, $a_n = \sum_{i=1}^{\infty} 2^{-i} i^{-n}$, $b_6 = H_{0,0,0,0,1,1}(\tfrac{1}{2})$, $b_7 = H_{0,0,0,0,0,1,1}(\tfrac{1}{2})$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$, $\text{Cl}_n(\theta) = \text{ImLi}_n(e^{i\theta})$. $H_{i_1, i_2, \dots}(x)$ are the harmonic polylogarithms. The integrals f_j are defined as follows:

$$\begin{aligned}
f_1(i, j, k) &= \int_1^9 ds D_1^2(s) \left(s - \frac{9}{5} \right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) , \\
f_2(i, j, k) &= \int_1^9 ds D_1(s) \text{Re} \left(\sqrt{3} D_2(s) \right) \left(s - \frac{9}{5} \right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) , \quad (26) \\
D_1(s) &= \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K \left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3} \right) , \\
D_2(s) &= \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K \left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3} \right) ;
\end{aligned}$$

$K(x)$ is the complete elliptic integral of the first kind. Note that $D_1(s) = 2J_2^{(1,9)}(s)$, with $J_2^{(1,9)}$ defined in Eq.(A.12) of Ref. [26]. The integrals $f_1(0, 0, 0)$ and $f_2(0, 0, 0)$ were studied in Ref. [6]. The

constants A_3 , B_3 and C_3 , defined in Ref. [6], admit the hypergeometric representations:

$$A_3 = \int_0^1 dx \frac{K_c(x)K_c(1-x)}{\sqrt{1-x}} = \frac{2\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3 \left(\begin{smallmatrix} \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2} \\ \frac{5}{6}, \frac{5}{6}, \frac{2}{3} \end{smallmatrix}; 1 \right) - \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3 \left(\begin{smallmatrix} \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6} \\ \frac{7}{6}, \frac{7}{6}, \frac{4}{3} \end{smallmatrix}; 1 \right) \right), \quad (27)$$

$$B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = \frac{4\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3 \left(\begin{smallmatrix} \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2} \\ \frac{5}{6}, \frac{5}{6}, \frac{2}{3} \end{smallmatrix}; 1 \right) + \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3 \left(\begin{smallmatrix} \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6} \\ \frac{7}{6}, \frac{7}{6}, \frac{4}{3} \end{smallmatrix}; 1 \right) \right), \quad (28)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = \frac{486\pi^2}{1925} {}_7F_6 \left(\begin{smallmatrix} \frac{7}{4}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{3}{2}, \frac{3}{2} \\ \frac{3}{4}, 1, \frac{7}{6}, \frac{11}{6}, \frac{13}{6}, \frac{17}{6} \end{smallmatrix}; 1 \right), \quad (29)$$

$$K_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1 \left(\begin{smallmatrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{smallmatrix}; x \right), \quad E_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1 \left(\begin{smallmatrix} \frac{1}{3}, -\frac{1}{3} \\ 1 \end{smallmatrix}; x \right). \quad (30)$$

A_3 appears only in the coefficients of the ϵ -expansion of master integrals, and cancels out in the diagram contributions. Fig.3 shows the fundamental elliptic master integrals which contains irreducible combinations of B_3 , C_3 and $f_m(i, j, k)$.

The analytical fits of V_{6b} , V_{6a} , V_{7b} , V_{7i} and the master integrals involved needed PSLQ runs with basis of ~ 500 elements calculated with 9600 digits of precision. The multi-pair parallel version [24] of the PSLQ algorithm has been essential to work out these difficult analytical fits in reasonable times.

The method used for the computation of the master integrals with precisions up to 9600 digits is essentially based on the difference equation method [1,2] and the differential equation method [27–29]. This method and the procedures used for the extraction of g -2 contribution, renormalization, reduction to master integrals, generation and numerical solution of systems of difference and differential equations, (all based on upgrades of the program **SYS** of Ref. [1]) will be thoroughly described elsewhere.

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-1.9122457649264455741526471674398300540608733906587253451713298480060
 384439806517061427608927000363158375584153314732700563785149128545391
 9028043270502738223043455789570455627293099412966997602777822115784720
 3390641519081665270979708674381150121551479722743221642734319279759586
 0740500578373849607018743283140248380251922494607422985589304635061404
 9225266343109442400023563568812806206454940132249775943004292888367617
 4889923691518087808698970526357853375377696411702453619601349757449436
 1268486175162606832387186747303831505962741878015305514879400536977798
 3694642786843269184311758895811597435669504330483490736134265864995311
 6387811743475385423488364085584441882237217456706871041823307430517443
 0557394596117155085896114899526126606124699407311840392747234002346496
 953173548258481799822409737371077365740464513521123091242528111372153
 0215445372101481112115984897088422327987972048420144512282845151658523
 6561786594592600991733031721302865467212345340500349104700728924487200
 6160442613254490690004319151982300474881814943110384953782994062967586
 7875385249781946989793132162197975750676701142904897962085050785592...

Table 1: First 1100 digits of $a_e^{(4)}$.

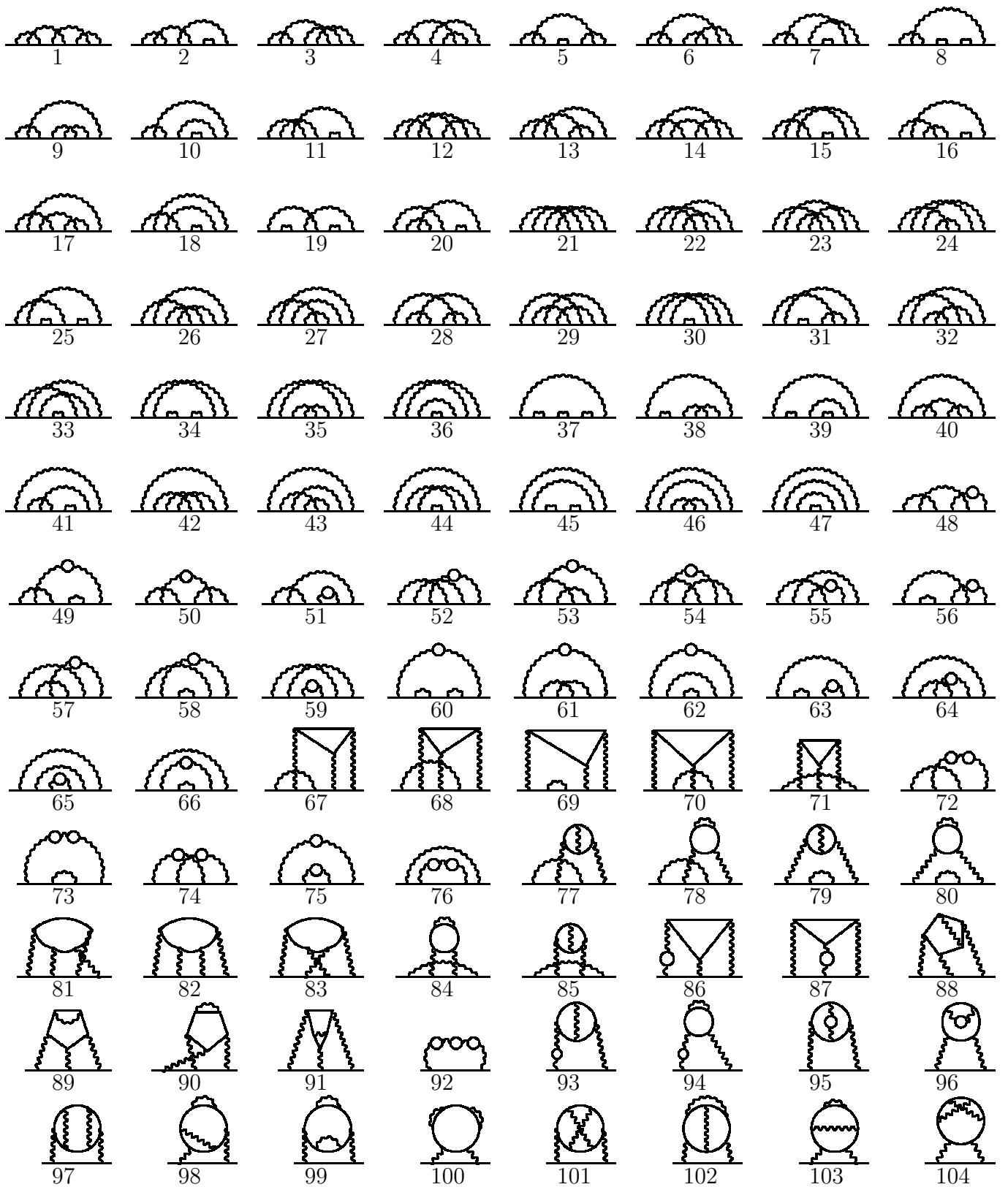


Figure 1: The 4-loop self-mass diagrams.

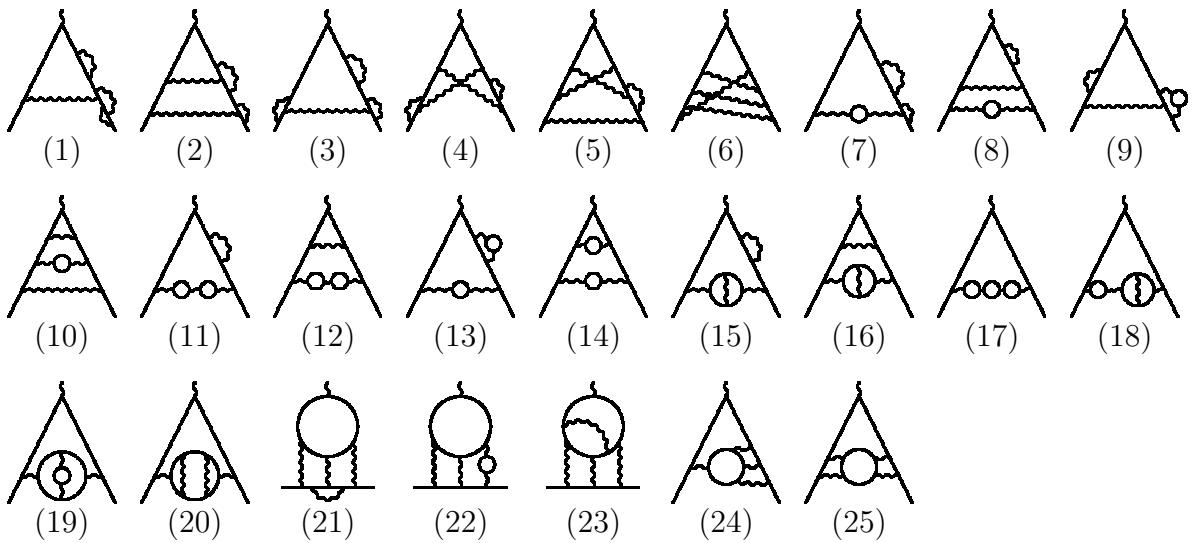


Figure 2: Examples of vertex diagrams belonging to the 25 gauge-invariant sets. The number indicates the gauge-invariant set to which the diagram belongs. In the case of the sets 1-16, 24,25, the other diagrams of each set can be obtained by permuting separately the vertices on the left and right side of the main electron line, and considering also the mirror images of the diagrams; in the sets containing diagrams with vacuum polarization insertions, one must also move the vacuum polarization insertion to each internal photon line. In the sets containing light-light diagrams, one must also consider the permutations of the vertices of the electron loop.

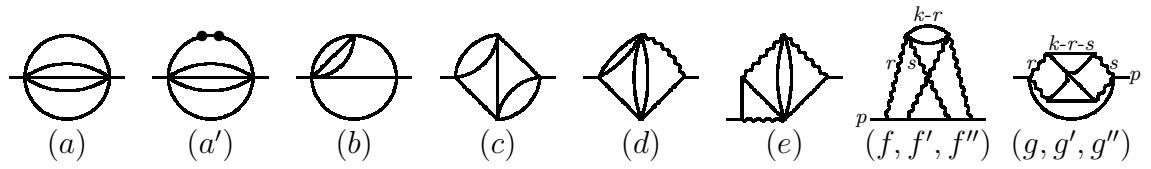


Figure 3: Minimal set of master integrals which contain all the elliptic constants. The double dot in (a') means that denominator is raised to the power three. (f, f', f'') and (g, g', g'') have numerators respectively equal to $(1, p.k, (p.k)^2)$.

1	- 1.971075616835818943645699655337264406980
2	- 0.142487379799872157235945291684857370994
3	- 0.621921063535072522104091223479317643540
4	1.086698394475818687601961404690600972373
5	- 1.040542410012582012539438620994249955094
6	0.512462047967986870479954030909194465565
7	0.690448347591261501528101600354802517732
8	- 0.056336090170533315910959439910250595939
9	0.409217028479188586590553833614638435425
10	0.374357934811899949081953855414943578759
11	- 0.091305840068696773426479566945788826481
12	0.017853686549808578110691748056565649168
13	- 0.034179376078562729210191880996726218580
14	0.006504148381814640990365761897425802288
15	- 0.572471862194781916152750849945181037311
16	0.151989599685819639625280516106513042070
17	0.000876865858889990697913748939713726165
18	0.015325282902013380844497471345160318673
19	0.011130913987517388830956500920570148123
20	0.049513202559526235110472234651204851710
21	- 1.138822876459974505563154431181111707424
22	0.598842072031421820464649513201747727836
23	0.822284485811034346719894048799598422606
24	- 0.872657392077131517978401982381415610384
25	- 0.117949868787420797062780493486346339829

Table 2: Contribution to $a_e^{(4)}$ of the 25 gauge-invariant sets of Fig.2.

T_0	9.515906781243876151283558690966098373
T_2	1915.31064825399777888130354499120276542
T_3	-3485.275086789599708317057778907752410742
T_4	3504.090225594272699233395974800847330934
T_5	-725.569913602974274507866288615667084989
T_6	1381.628304197738147258897402093908402776
T_7	1692.786400388934476652564199811210670453
V_{4a}	-223.655742930151691157141102901111870825
V_{6a}	14.029138087062071859189974573196626739
V_{6b}	842.150210099809624937684343426149287354
V_{7b}	463.951882993580804359224932846794527895
W_{6b}	-1560.934864680405790411777238139658336036
W_{7b}	-1024.004093725178841133583200254534168436
E_{4a}	-856.605968292200108497784694038000040595
E_{5a}	601.136193120690233763409588135510244820
E_{6a}	-457.790342894702531083496436277945999328
E_{6b}	-89.049936952630079330356943951138211140
E_{7a}	548.453177743013238987339022298522918205
E_{7b}	-2145.946406417837479874008380333397996999
U	- 132.027597619729495491707871522090745221
C_{81a}	116.694585791186600526332510987652818034
C_{81b}	- 8.748320323814631572671010051472284815
C_{81c}	- 0.236085277120339887503638687666535683
C_{83a}	2.771191986145520146810618363218497216
C_{83b}	- 0.807847353263827557176395243854200179
C_{83c}	- 0.434702618543809180642530601495074086

Table 3: Numerical values of the constants of Eq.7 and Eq.25.